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FREE OSCILLATIONS OF LINEAR SYSTEMS
WITH VARIABLE PARAMETERS

By

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SVOBODNYYE KOLEBANIYA LINEYNYKH SISTEM S PEREMENNYMI PARAMETRAMI

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ABSTRACT: ~~This~~ THE book presents the principles of the theory of oscillations of linear systems with variable parameters, gives the results of investigations of the stability of oscillations, and examines the means for analyzing oscillations in a finite time interval.

This book is intended for scientific workers and engineers who encounter, during their work, questions of the theory of oscillations and, in particular, for specialists in the calculation of the flight dynamics of flight vehicles and specialists in automatic control.

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ë in Russian, transliterate as yë or ë. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH

DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}

Russian	English
rot	curl
lg	log

This book presents the principles of the theory of oscillations of linear systems with variable parameters, gives the results of investigations of the stability of oscillations, and examines the means for analyzing oscillations in a finite time interval.

This book is intended for scientific workers and engineers who encounter, during their work, questions of the theory of oscillations and, in particular, for specialists in the calculation of the flight dynamics of flight vehicles and specialists in automatic control.

PREFACE

Questions of analysis of free oscillations of linear systems with variable parameters in our time have interested a wide circle of specialists working in different regions of technology, astronomy and physics. For successful and fast resolution of all possible problems connected with the theory of oscillations of systems of this class, methods of analysis of oscillations are necessary which are convenient in application and effective. These methods, furthermore, have to be accessible to a specialist whose knowledge in the region of theory of oscillations and in neighboring areas of mathematics corresponds to a program of instruction in a technical institute. The absence of a literary source, in which there would be presented methods satisfying these requirements and embracing a sufficiently wide range of questions of theory of oscillations, impelled the author to write this book.

In the book the reader will not find wide illumination of different methods of analysis of oscillations. On the contrary, all work is connected to a single method of research. In this method there is synthesized some of the latest research of native and foreign scientests and theoretical constructions connected with canonical expansions of solution of equation of free oscillations. The limited size of the work did not allow the author to bring it to the state where the problem of establishing necessary methods of analysis of oscillations, in light of the above-mentioned requirements, could have been considered solved. Nonetheless, presented in this book, methods of solution of a number of problems, based on the offered method, satisfy these requirements; however, a large part of the given theoretical constructions can be the beginning for further search.

The book consists of an introduction and eight chapters. The introduction

introduces the reader to a circle of ideas assumed as the basis of the method of research utilized in the book. It is designed for a reader familiar with the basic ideas of the theory of stability of systems with variable parameters, and can be passed during reading of book by unprepared reader. In the first chapter are presented basic ideas and definitions from the theory of oscillations and the theory of differential equations utilized during further research. In this chapter are determined ideas of an equation of free oscillations and equivalent systems of equations. In the second chapter is expounded the theory of canonical expansions of solution of an equation of free oscillations. These expansions allow us to replace equations of free oscillations by systems of linear, homogeneous differential equations of special form, whose elements of theory are expounded in the third chapter. In the fourth chapter are considered questions connected with analysis of free oscillations in a finite interval of time. In the fifth and sixth chapters are investigated asymptotic properties of free oscillations. The seventh chapter contains additional research in linear systems with periodically variable parameters; and the eighth chapter, additional research in linear systems, whose equations of free oscillations belong to a certain special class of equations, the particular form of which are equations with polynomial coefficients.

The book is designed for scientists specialized in the region of theory of oscillations and for engineers whose activity is connected with analysis of dynamics of technical systems. In order to make the book accessible for engineers, the author has avoided the use of mathematical ideas (besides those known from a course of higher mathematics in technical colleges), whose explanation the reader would have to seek in other literary sources; ideas not finding reflection in the mentioned courses but used in this work are explained in foot notes or in those places of the basic text where they are first encountered. With the same aim, each chapter in which are expounded these or other methods of analysis is closed with an example including detailed calculations.

In the book, as a rule, are not considered proofs of theorems and assumptions which are used in this work but published earlier in the press. In all such cases are shown references to literature. Part of the results of the research done by the author and given in the present work was published in 1959-60 in the articles "Theory of Free Oscillations of Linear Systems with Variable Parameters" [1] and "Canonical Conversions of Equations of Free Oscillations of Linear Systems with Variable

Parameters and Their Application to Analysis of Oscillations" [2].

The author expresses gratitude to academician B. N. Petrov, who has rendered essential help during the writing of this book. The author also is very grateful to A. I. Averbukh, S. M. Alfërov, L. N. Bol'shev, G. N. Duboshin, A. A. Lebedev, A. M. Letov, G. N. Sveshnikov and V. V. Solodovnikov for their counsel and critical remarks.

F. A. Mikhaylov.

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INTRODUCTION

With the development of new technology, interest has increased toward the theory of dynamic processes described by linear differential equations with variable coefficients and, in particular, toward the theory of free oscillations of linear dynamic systems described, as is known, by linear homogeneous equations.

In this work we examine a very wide-spread type of free oscillations of linear dynamic systems, whose characteristic peculiarity consists of the fact that the system of differential equations describing the oscillations can be reduced to one, linear, homogeneous equation of n-th order with continuous differentiable coefficients, solved relative to senior derivative and containing, as an independent variable, time t and, as an unknown variable interesting the researcher, coordinate x (or any other characteristic of the state of the system),

$$\frac{d^n x}{dt^n} + b_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + b_n(t)x = 0. \tag{0.1}$$

As particular cases in the work are considered free oscillations, presented by equation (0.1) with coefficients possessing, besides those shown, also certain other properties.

Recently was published a series of works based on the theory of free oscillations of linear systems with variable parameters, in which was considered the shown type of free oscillations. Not touching upon the articles of the author [1, 2], the results of which are presented in the present work, we will pause on two researches dedicated to this type of oscillations and belonging to I. P. Ginzburg and P. Grensted. The work of I. P. Ginzburg [3] contains a definition of sufficient conditions of limitedness of solution of equation (0.1) and its derivative from first to n-1st inclusively.

Results obtained by its author, in the case of a second order equation

$$\frac{d^2x}{dt^2}+b_1(t)\frac{dx}{dt}+b_2(t)x=0 \qquad (0.2)$$

coincide with the following known by the works of other authors:

For limitedness of solution of equation (0.2) and its first derivative, it is sufficient that, starting from a certain moment of time t = T for certain positive constants D and d during all values of t, there be executed conditions

$$D>b_2>d \qquad (0.3)$$

and one of the conditions

$$b_1+\frac{1}{2}\frac{\dot{b}_2}{b_2}>0 \qquad (0.4)$$

or

$$b_1>d>0, \quad \dot{b}_2\leqslant 0. \qquad (0.5)$$

The author of the second work, P. Grensted [4], sets as his goal the determination of rational conditions which are rational for technical application and which are approximations, without definite measure, to necessary and sufficient conditions of damping of free oscillations, i.e., to conditions at which solution of equation (0.1) is a vanishing function. As a result of conditional reasonings the author obtains such conditions. In reference to equation (0.2), its results may be formulated in the following form.

Condition (0.4) executed from certain moment of time t = T, is an approximation to the necessary and sufficient condition of damping of the free oscillations presented by equation (0.2).

Works [3] and [4], absolutely, present interest and can be used as applications. However, results, obtained in them, essentially, carry a very particular character. Thus, in the consided example, illustrating result of work [3], condition (0.3) is a strong limitation, because of which it is impossible to reveal limitedness of solution even in the case of such a simple equation as Airy equation

$$\ddot{x}+tx=0. \qquad (0.6)$$

In the example illustrating results of work [4], a strong limitation, narrowing region of application of results, is the requirement of fulfillment of inequality (0.4) during all values of t starting from a certain t = T, for equation

$$\ddot{x}+b_2(t)x=0 \qquad (0.7)$$

This requirement leads to requirement of monotonicity of function $B_2(t)$ in interval (T, ∞).

For analysis of considered type of free oscillations, there can be used results of much research carried out in reference to broader classes of free oscillations. Such a possibility, in particular, follows from equivalence of equation (0.1) to system of equations

$$\left.\begin{array}{l} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \cdot\ \cdot\ \cdot\ \cdot \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -b_n x_1 - b_{n-1} x_2 - \ldots - b_1 x_n, \end{array}\right| \qquad (0.8)$$

to which is brought equation (0.1) by substitution

$$x = x_1, \quad \frac{dx}{dt} = x_2, \quad \frac{d^2x}{dt^2} = x_3, \ldots, \frac{d^{n-1}x}{dt^{n-1}} = x_n. \qquad (0.9)$$

Because of this equivalence, sufficient conditions of limitedness of solution of equation (0.1) and its derivative from first to n-1st inclusively are transformed to sufficient conditions of stability of zero solution of system of equation (0.8). The idea of stability of zero solution corresponds to the following determination.

Zero solution of system of differential equations (0.8) is stable if it is possible to indicate such a real number T at which any arbitrarily assigned number ε corresponds to positive number λ so that during any real values of ξ_s magnitudes of x_s for $t = T$, satisfying conditions

$$|\xi_s| \leqslant \lambda,$$

of inequality

$$|x_s| < \varepsilon \quad (s = 1, 2, \ldots, n)$$

will be fulfilled for any value of $t \geqq T$ [5].

Using shown equivalence, it is possible to apply results obtained by different authors for systems of linear differential equations to solution of the problem considered in the work of Ginzburg.

Usually, during the study of questions interesting us, the system of linear differential equations is considered in the form

$$\left.\begin{array}{l} \dot{x}_1 = a_{11} x_1 + \ldots + a_{1n} x_n, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dot{x}_n = a_{1n} x_1 + \ldots + a_{nn} x_n, \end{array}\right\} \qquad (0.10)$$

where $a_{ij} = a_{ij}(t)$ $(i, j = 1, \ldots, n)$ are real, continuous functions. Above-mentioned definition of the idea of stability of zero solution extends to a case of such systems

without any change.

The problem of determining sufficient conditions of stability of zero solution of system (0.10), a particular case of which is system (0.8), was considered by many researchers and in their works they obtained partial solution. Not troubling the reader by a survey of results of all basic research on this question, we will characterize only two methods of solution of this problem, which by the number of works dedicated to them and by the effectiveness of results obtained in certain of them, can be considered basic.

The first method is based on the widely known theorem of A. M. Lyapunov about stability of motion, which in our case it is possible to formulate as the theorem of stability of zero solution of system of equations (0.10), and consists of detecting real functions $V(t, x_1, \ldots, x_n)$ and $W(x_1, \ldots, x_n)$, possessing certain individual properties.

Theorem of Lyapunov [6]. Zero solution of system of differential equations (0.10) is stable if there exist such functions $V(t, x_1, \ldots, x_n)$ and $W(x_1, \ldots, x_n)$, which fulfill the following conditions:

a) $W(x_1, \ldots, x_n) > 0$, if at least for one i magnitude of x_i is different than zero;

b) there exists such a value $t = T$, that during $t \geq T$ there occurs inequality

$$V(t, x_1, \ldots, x_n) - W(x_1, \ldots, x_n) \geq 0;$$

c)
$$\frac{\partial V}{\partial t} + \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} (a_{i1}x_1 + \ldots + a_{in}x_n) \leq 0.$$

Given theorem connects the fact of stability of zero solution of system (0.10) with the fact of the existence of certain functions of variables $t, x_1, \ldots, x_n$. However, not from formulation of the theorem nor from proof given by Lyapunov does there ensue any method for finding these functions. Finding functions V and W, satisfying conditions of this theorem, depends on the skill and inventiveness of the researcher [5], and no general method for solution of this problem, up to now, has been found.

In 1937 K. P. Persidskiy [7] proved the reversibility of the theorem of Lyapunov, noting that if zero solution of system (0.10) with real continuous coefficients is stable, then exist functions $V(t, x_1, \ldots, x_n)$ and $W(x_1, \ldots, x_n)$, possessing properties shown in conditions of the theorem of Lyapunov. K. P. Persidskiy showed also that function $V(t, x_1, \ldots, x_n)$ both in the theorem of Lyapunov and in converse theorem

can be considered in the form of quadratic form of variables $x_1, \ldots, x_n$ with coefficients depending on t.

Theorem of Lyapunov is the initial point of the first method for determining sufficient conditions of stability of zero solution of system (0.10). Further lines of development of the method are connected with the method of constructing these functions. At present there is known a rather large number of different methods of constructing them, some of which [5, 8, 9, 10] have obtained wide acknowledgement. In accordance with the results of the research of Persidskiy in all methods function V is constructed in quadratic form.

The initial point of the second method of solving the problem about determining sufficient conditions of zero solution of system (0.10) is the theorem of A. Wintner [11], published in 1946.

Theorem of Wintner. If the coefficients of the system of differential equations (0.10) are continuous, then for its real solution during any limited initial values of variables $x_1, \ldots, x_n$, the following inequality is valid:

$$|x(t)| \leqslant |x(0)| \exp \int_0^t \max_{|x|=1} \sum_{i,j=1}^n a_{ij} x_i x_j \, dt. \tag{0.11}$$

where

$$|x| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

Because of this theorem for stability of zero solution of system (0.10) (we consider that coefficients of equations satisfy conditions of theorem), it is sufficient that the following inequality be fulfilled:

$$\overline{\lim_{t\to\infty}} \int_0^t \max_{|x|=1} \sum_{i,j=1}^n a_{ij} x_i x_j \, dt < \infty. \tag{0.12}$$

The Wintner theorem in 1948 was generalized by T. Vazhevskiy [12], who, in particular, paid attention to equality

$$\max_{|x|=1} \sum_{i,j=1}^n a_{ij} x_i x_j = \mu_n,$$

where symbol μ_n is designated the biggest root of equation[1]

---

[1] All roots of this equation are real, see § 6, Ch. III.

$$\begin{vmatrix} a_{11}-\mu & \frac{a_{12}+a_{21}}{2} & \dots & \frac{a_{1n}+a_{n1}}{2} \\ \frac{a_{12}+a_{21}}{2} & a_{22}-\mu & \dots & \frac{a_{2n}+a_{n2}}{2} \\ \dots & \dots & \dots & \dots \\ \frac{a_{1n}+a_{n1}}{2} & \frac{a_{2n}+a_{n2}}{2} & \dots & a_{nn}-\mu \end{vmatrix}=0$$

Using this equality, it is possible to record sufficient condition of stability (0.12) in the form

$$\overline{\lim_{t\to\infty}}\int_0^t \mu_n\,dt<\infty. \tag{0.13}$$

Thus, in the theorems of Wintner and Vazhevskiy there is contained a fully defined constructive solution of the considered problem. However, effectivenss of the sufficient conditions of stability, obtained on the basis of its application, essentially depends on the form of system of equations (0.10) and, in a number of cases, can be very low.

In 1950 A. D. Gorbunov [13] proved a theorem, on the basis of which can be obtained sufficient conditions of stability more effective than condition (0.13).

Theorem of Gorbunov. If coefficients of differential equations (0.10) are continuous, quadratic form

$$G(x_1,\dots,x_n)=\sum_{i,j=1}^{n} g_{ij}(t)x_ix_j \quad (g_{ij}=g_{ji})- \tag{0.14}$$

during $t>0$ is positively determined (see § 6, Ch. III) with continuous differentiable coefficients and

$$H(x_1,\dots,x_n)=\sum_{i,j=1}^{n} h_{ij}(t)x_ix_j-$$

its derivative, because of differential equations (0.10), $\mu_1(t)$ is the least, and $\mu_n(t)$ is the biggest roots of the equation[1]

$$\begin{vmatrix} h_{11}-2\mu g_{11}, & h_{12}-2\mu g_{12}, & \dots, & h_{1n}-2\mu g_{1n} \\ h_{21}-2\mu g_{21}, & h_{22}-2\mu g_{22}, & \dots, & h_{2n}-2\mu g_{2n} \\ \dots & \dots & \dots & \dots \\ h_{n1}-2\mu g_{n1}, & h_{n2}-2\mu g_{n2}, & \dots, & h_{nn}-2\mu g_{nn} \end{vmatrix}=0, \tag{0.15}$$

then during any real limited initial values of variables $x_1, \dots, x_n$, the following inequalities are valid

---

[1]All roots of this equations are real.

$$\sqrt{G[x_1(0), \ldots, x_n(0)] \frac{\Delta_{n-1}^{(i)}}{\Delta_n}} \exp \int_0^t \mu_1 \, dt \leqslant |x_i| \leqslant$$

$$\leqslant \sqrt{G[x_1(0), \ldots, x_n(0)] \frac{\Delta_{n-1}^{(i)}}{\Delta_n}} \exp \int_0^t \mu_n \, dt \qquad (0.16)$$

$$(i = 1, \ldots, n),$$

where

$$\Delta_n = \begin{vmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ g_{n1} & g_{n2} & \cdots & g_{nn} \end{vmatrix};$$

$\Delta_{n-1}^{(i)}$ is the minor obtained from this determinant by obliterating i-th column and i-th line.

Because of this theorem for stability of zero solution of system (0.10), it is sufficient that the following inequalities are fulfilled:

$$\overline{\lim_{t \to \infty}} \left(\frac{\Delta_{n-1}^{(i)}}{\Delta_n} \exp \int_0^t \mu_n \, dt \right) < \infty \qquad (0.17)$$

$$(i = 1, \ldots, n).$$

At $G = x_1^2 + \ldots + x_n^2$ these inequalities are turned into inequality (0.13).

The theorem of Gorbunov essentially strengthens the second method of solving the problem of determining sufficient conditions of stability of zero solution of system (0.10). However, from this theorem it is impossible to extract any indications about rational selection of form $G(x_1, \ldots, x_n)$, and, consequently, here, just as in the first method, success depends on the skill and inventiveness of the researcher. The second method does not have among its assets such extensive developments as the first (usually is considered form $G = x_1^2 + \ldots + x_n^2$, see, for instance, article of Gorbunov [14]); nonetheless the method is rich in its potential possibilities.

Let us consider now a particular case of system (0.10) — system (0.8) and we will observe to what results leads solution of the problem about stability of its zero solution by the first and second methods.

As an example of application of the first method to the solution of this problem is the quoted-above work of I. P. Ginzburg. The author of this work attempted to obtain, as far as possible, more effective conditions of stability of zero solution; his method of construction of function $V(t, x_1, \ldots, x_n) = G(x_1, \ldots, x_n)$ is now one of the most rational.

Constructed by Ginzburg, function V possesses the following interesting properties: during constant coefficients of system (0.8), derivative of this function, because of differential equations of system, identically turns into zero, but requirement of positive definitiveness of form $G(x_1, \ldots, x_n)$ leads to conditions of stability Herwitz-Routh. In accordance with this property, sufficient conditions of stability, defined by Ginzburg for a case of variables of coefficients of system (0.8), approach, as closely as possible, to the necessary conditions of stability if coefficients of system (0.8) become sufficiently close to those formed (on the question of mathematical meaning of the idea of proximity, used here, we will not pause).

Now we will define sufficient conditions of stability of zero solution of system (0.8) by the second method, using system (0.8) in its initial form. We will be limited by system of the second order, corresponding to equation (0.2), and we will assume the form of G to be $G = x_1^2 + \ldots + x_n^2$.

Equation (0.15) here will take the form

$$\begin{vmatrix} -\mu & \frac{1-b_2}{2} \\ \frac{1-b_2}{2} & -b_1-\mu \end{vmatrix} = 0, \tag{0.18}$$

whence it follows

$$\mu_2 = -\frac{b_1}{2} + \frac{\sqrt{b_1^2+(1-b_2)^2}}{2}.$$

Condition (0.17) is fulfilled if

$$\int_0^\infty \left[-b_1+\sqrt{b_1^2+(1-b_2)^2}\right] dt < \infty. \tag{0.19}$$

In order to estimate effectiveness of this sufficient condition, we will assume that coefficients b_1 and b_2 are constant. Then inequality (0.19) will be executed only in the case

$$b_1 > 0, \quad b_2 = 1. \tag{0.20}$$

Obviously, sufficient conditions of stability (0.20) are far from conditions of Herwitz-Routh. Consequently, effectiveness of condition (0.20) and, all the more so, effectiveness of condition (0.19) for variable coefficients of system (0.8) are very low.

Generalizing the statement, it is possible to note that application of the most well-known methods of determining sufficient conditions of stability of zero solution

of system (0.10), by the second method, to the problem of determining sufficient conditions of limitedness of solution and derivatives of solution of equation (0.1), with the help of transition to system (0.8), does not lead to more effective results, as compared to results obtained by I. P. Ginzburg.

It would have been possible to continue survey of known results, using other forms of transition from equation (0.1) to equivalent system of equations, and also considering other problems besides the problem considered by Ginzburg. However, it is doubtful whether this would be expedient, since, first, published literature on the considered questions contains very few effective (in practice) results and, secondly, considerations, which are assumed in the basis of the research given in the present work, are presented in a fully intelligible manner. These considerations are presented below.

From a short survey of methods of solving the problem of determining sufficient conditions of stability, it is possible to conclude certain results.

First of all, let us note that in seeking the most rational path for further searches, one should give preference to the second method since in this case on derivative of function $G(x_1, \ldots, x_n)$ are put weaker limitations (its sign-alternating is allowed) than on derivative of function $V(t, x_1, \ldots, x_n)$. Because of this, it is possible to expect that for determination of equally effective, sufficient conditions of stability in the second method will be demanded simpler structures of functions $G(x_1, \ldots, x_n)$ than structure of functions $V(t, x_1, \ldots, x_n)$ in the first method.

Further, if one were to return to the above-considered example illustrating the second method, then it is easy to notice that at b_1 = const, b_2 = const, $b_1^2 > 4b_2$ there can be obtained sufficient conditions of stability in the form of Herwitz-Routh conditions, after using substitution

$$\left.\begin{aligned} x_1 &= y_1 + y_2, \\ x_2 &= \lambda_1 y_1 + \lambda_2 y_2, \end{aligned}\right\} \qquad (0.21)$$

where λ_1 and λ_2 are roots of equation

$$\lambda^2 + b_1\lambda + b_2 = 0.$$

Actually, applying this substitution to system

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -b_2 x_1 - b_1 x_2, \end{aligned}\right\}$$

can be obtained system

$$\left.\begin{aligned}\dot{y}_1 &= \lambda_1 y_1,\\ \dot{y}_2 &= \lambda_2 y_2.\end{aligned}\right\} \tag{0.22}$$

For this system

$$\nu_2 = \max(\mathrm{Re}\,\lambda_1,\ \mathrm{Re}\,\lambda_2). \tag{0.23}$$

Condition (0.13) is turned into condition

$$\nu_2 = \max(\mathrm{Re}\,\lambda_1,\ \mathrm{Re}\,\lambda_2) < 0, \tag{0.24}$$

which is executed during fulfillment of Herwitz-Routh conditions.

Because of equations (0.21) from sufficient conditions of stability of zero solution of system (0.22) follow sufficient conditions of stability of zero resolutio of initial system.

From this example it follows that change of variables can effectively be used as an additional method during research of stability by the second method. Furthermore, given example allows us to conclude another logical result if one were to otherwise interpret method of obtaining condition (0.24).

Really, because of equation (0.21) form $|y_1|^2 + |y_2|^2$ corresponds to a certain positive definite form $G(x_1, x_2)$, namely:

$$G(x_1, x_2) = |y_1|^2 + |y_2|^2 = \left|\frac{x_1\lambda_2 - x_2}{\lambda_2 - \lambda_1}\right|^2 + \left|\frac{x_1\lambda_1 - x_2}{\lambda_1 - \lambda_2}\right|^2.$$

If roots λ_1 and λ_2 are real, then, passing from roots to coefficients b_1 and b_2, we will obtain

$$G(x_1, x_2) = \frac{(b_1^2 - 2b_2)x_1^2 + 2b_1x_1x_2 + 2x_2^2}{b_1^2 - 4b_2}. \tag{0.25}$$

If roots λ_1 and λ_2 are complex, then form of $G(x_1, x_2)$ it is possible to present in the form

$$G(x_1, x_2) = 2\frac{b_2x_1^2 + b_1x_1x_2 + x_2^2}{4b_2 - b_1^2}. \tag{0.26}$$

It is natural that in both cases form of $G(x_1, x_2)$ is positively definite.

Form of $H(x_1, x_2)$ in the first case will take the form

$$H(x_1, x_2) = -\frac{2b_1b_2x_1^2 + 8b_2x_1x_2 + 2b_1x_2^2}{b_1^2 - 4b_2}.$$

In the second case, the form

$$H(x_1, x_2) = -\frac{2(b_1b_2x_1^2 + b_1^2x_1x_2 + b_1x_2^2)}{4b_2 - b_1^2}.$$

Finding root μ_2 of equation (0.15), it is easy to establish that in both cases condition

$$\mu_2 < 0$$

is executed, if the Herwitz-Routh conditions are executed.

Thus, it is possible to prove sufficiency of Herwitz-Routh conditions for stability of zero solution of the considered system without transition to variables y_1 and y_2, after selecting form $G(x_1, x_2)$ according to formula (0.25) or (0.26). Selection of the type of utilized form depends on character of the root. This conclusion may be widely used on systems of the highest orders (with constant coefficients). Apparently, even for systems with variable coefficients the character of the roots of certain equations should be considered during research of stability.

Now we will pay attention to the circumstance that substitution (0.21) leads the considered system to a system whose matrix of coefficients has a diagonal form. With this fact it is possible to connect high effectiveness of obtained results. Generalizing this observation, the conclusion can be made that expedient are such changes of variables which somehow bring the matrix of coefficients of the system nearer to a matrix of diagonal form.

At last, it is possible to note, while developing thought about change of variables, that only such changes of variables are expedient at which it is quite easy to judge properties of zero solutions of initial systems with respect to properties of zero solutions of converted systems.

These thoughts predetermined the direction of the research which the reader will encounter below. This research is not limited by questions of stability or limitedness of solution, and embrace a wider range of questions. Considered are the following problems:

a) determination of majorant and minorant estimations of particular solutions of equation of oscillations (0.1) in a given finite interval, and also majorant and minorant estimations of certain functions connected with solution and convenient for analysis of free oscillations ;

b) determination of approximate presentations of general and particular solutions of equation of oscillations ;

c) determination of conditions of stability of oscillations ;

d) determination of conditions of existence of fluctuating solutions of equation of oscillations ;

e) determination of asymptotic representations of general and particular

solutions of equation of oscillations.

All work is connected by unity of method of research, in the basis of which lie canonical expansions of solution of the equation of free oscillations.

We will explain the essence of the method in an example of the problem about determination of sufficient conditions of limitedness of solutions of equation of free oscillations of the second order [equation (0.2)].

We will present arbitrary solution of equation (0.2) $x(t)$ in the form of sum of functions $z_1(t)$ and $z_2(t)$, satisfying system of equations

$$\left.\begin{aligned} x &= z_1 + z_2, \\ \dot{x} &= \zeta_1 z_1 + \zeta_2 z_2, \end{aligned}\right\} \qquad (0.27)$$

where ζ_1 and ζ_2 are arbitrary, complex-valued, differentiable functions from t, unequal to each other during all sufficiently large values of t.

From equations (0.27) and (0.2) it is simple to obtain a system which variables z_1 and z_2 satisfy. It has the form

$$\left.\begin{aligned} \dot{z}_1 &= \left(\zeta_1 + \frac{\dot{\zeta}_1 + \zeta_1^2 + b_1\zeta_1 + b_2}{\zeta_2 - \zeta_1}\right) z_1 + \frac{\dot{\zeta}_2 + \zeta_2^2 + b_1\zeta_2 + b_2}{\zeta_2 - \zeta_1} z_2, \\ \dot{z}_2 &= \frac{\dot{\zeta}_1 + \zeta_1^2 + b_1\zeta_1 + b_2}{\zeta_1 - \zeta_2} z_1 + \left(\zeta_2 + \frac{\dot{\zeta}_2 + \zeta_2^2 + b_1\zeta_2 + b_2}{\zeta_1 - \zeta_2}\right) z_2. \end{aligned}\right\} \qquad (0.28)$$

If functions $\zeta_1(t)$ and $\zeta_2(t)$ satisfy condition

$$\zeta_{1,2}^2 + \dot{\zeta}_{1,2} + b_1\zeta_{1,2} + b_2 = 0, \qquad (0.29)$$

then matrix of coefficients of right side of system (0.28) is diagonal, and each of the equations of this system may be integrated separately. Designating initial conditions

$$z_1(t_0) = 1, \quad z_2(t_0) = 0$$

and

$$z_1(t_0) = 0, \quad z_2(t_0) = 1.$$

(where t_0 is sufficiently great), we will obtain two particular solutions of equation (0.2)

$$x^{(1)} = \exp\int_{t_0}^{t} \zeta_1\,dt, \quad x^{(2)} = \exp\int_{t_0}^{t} \zeta_2\,dt. \qquad (0.30)$$

Hence immediately follows solution of problem about determination of sufficient conditions of limitedness of solutions of equation (0.2): for limitedness of all solutions of this equation, limitedness of shown solutions is sufficient.

It is possible to prove, based on polar presentation of general solution of equation (0.2) [25], that there exist functions $\zeta_{1,2}(t)$ satisfying condition (0.29). But since these functions, in a one-to-one manner, correspond to solutions (0.30), the problem of their determination is just as difficult as the problem of determining

shown solutions.

However, it is possible to indicate a whole series of methods of selection of functions $\zeta_{1,2}(t)$, at which condition (0.29) is executed with the sign of not strict but approximate equality with this or another meaning of proximity. Let us assume that functions $\zeta_{1,2}(t)$ are selected thus. Then, assuming that they are substantial and using the result of Vazhevskiy, we will find that for limitedness of solutions of equation (0.2) fulfillment of condition (0.13) is sufficient, where μ_n is the biggest root of equation

$$\begin{vmatrix} \zeta_1 + \frac{\dot{\zeta}_1 + \zeta_1^2 + b_1\zeta_1 + b_2}{\zeta_2 - \zeta_1} - \mu & \frac{b_1 + \zeta_1 + \zeta_2}{2} + \frac{\dot{\zeta}_1 - \dot{\zeta}_2}{2(\zeta_1 - \zeta_2)} \\ \frac{b_1 + \zeta_1 + \zeta_2}{2} + \frac{\dot{\zeta}_1 - \dot{\zeta}_2}{2(\zeta_1 - \zeta_2)} & \zeta_2 + \frac{\dot{\zeta}_2 + \zeta_2^2 + b_1\zeta_2 + b_2}{\zeta_1 - \zeta_2} - \mu \end{vmatrix} = 0. \quad (0.31)$$

This sufficient condition during successful selection of functions $\zeta_{1,2}(t)$ may be very effective since during strict fulfillment of condition (0.29) it coincides with that necessary.

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CHAPTER I

LINEAR SYSTEMS WITH VARIABLE PARAMETERS AND EQUATIONS OF THEIR FREE OSCILLATIONS

§ 1. Dynamic Systems

In nature and technology there are wide spread physical systems whose state at an arbitrary fixed moment of time t, from a certain interval of observation (t_0, t_1), is fully determined by values of physical quantities (at this instant of time), presented mathematically by real variables

$$x_1, x_2, \ldots, x_n,$$

and fully determines motion of system in a subsequent interval of time. Such systems are called dynamic [15].

To dynamic systems, in particular, belong mechanical conservative systems [15]. In this case variables x_1, ..., x_n are generalized coordinates (i.e., a certain number of independent physical quantities, simply determining coordinates of all elements of system [16]) and their first derivatives with respect to t. Full system of generalized coordinates, in particular, can constitute k of certain coordinates of its elements. In this case their derivatives with respect to t are corresponding speeds of displacement of these elements.

To dynamic systems also belong electrical systems composed of passive circuits, i.e.. systems consisting of a certain number of electrically connected capacitors, conductors, and chokes. Such systems are electrical analogs of mechanical systems considered by theoretical mechanics. Role of generalized coordinates here is executed by k of independent physical quantities determining distribution of charges in system, in particular charges of a certain number of capacitors. In the last case,

derivatives of generalized coordinates are the intensities currents flowing through capacitors.

In different regions of technology are encountered dynamic systems very varied in their physical nature. Frequently, they constitute a combination of interacting mechanical, electrical, and magnetic elements and contain internal sources of energy, essentially affecting their properties. During determination of physical quantities $x_1, \ldots, x_n$, characterizing the state of such systems, in many cases is expedient an essential idealization of the laws of work of certain elements of the system or combinations of elements, without which mathematical description of the system can be unnecessarily complicated and difficult for analysis. Such idealization frequently is connected with disregard of small masses and inductances and hypothetical acknowledgement of directivity of interaction of certain elements of system, i.e., acknowledgement of direct influence of one element on another and disregard of influence of the latter on the first.

The simplest example of a dynamic system is a mechanical system consisting of a body of mass m, a rod and two springs, united so that the body under the action of the springs can accomplish horizontal motions along the rod (Fig. 1). State of such a system is determined by displacement x of center of gravity of body about position of equilibrium and speed of displacement $\dot{x}$.

The simplest example of a dynamic system is also an electrical oscillation circuit constituting of a closed circuit composed of capacity C, resistance R, and inductance L (Fig. 2). State of such a system is determined by charge of capacitor and current intensity in circuit.

Examples of more complicated dynamic systems under certain conditions can be an aircraft flying in calm atmosphere, a vacuum-tube oscillator, and an elastic mechanical system with dynamic extinguisher of oscillations.

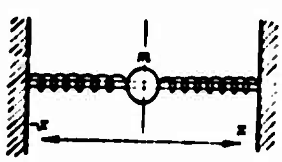

Fig. 1. The simplest mechanical dynamic system.

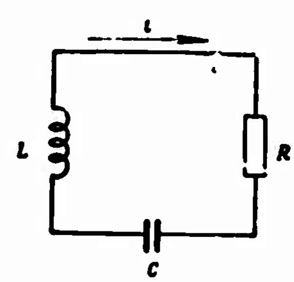

Fig. 2. Electrical oscillation circuit

An aircraft flying in a calm atmosphere is a dynamic system if the speed of its flight exceeds the speed of sound[1] and rudders, elevators, and ailerons are secured or controlled by means of an automatic pilot. In the first, simpler case (secured rudders, elevators, and ailerons) the state of the system is determined by linear and angular coordinates of the aircraft, characterizing its position relative to land (all six coordinates) and the first derivatives of these coordinates (corresponding linear and angular velocities). Thus, the number of magnitudes determining state of system is equal to twelve. In spite of the fact that this number is very great, it cannot be decreased (in examining general case of motion) without disturbance of correct presentation of system. Moreover, even the presented treatment of the system is based on a series of simplifying assumptions concerning interaction of the aircraft with air flow, and is practically correct only in that case when thrust of motors and consumption of fuel are fully determined by the above-mentioned magnitudes.

A vacuum-tube oscillator of the simplest form consists of two batteries, an electron tube, a choke, and an oscillation circuit inductively connected with it (Fig. 3). During the study of this system, usually, we will disregard drops of voltage in choke and inductive winding of circuit, called grid current [19]. Under this condition, the state of the system is determined by charge of capacitor and current intensity passing through inductive winding of oscillation circuit.

The simplest elastic mechanical system with a dynamic extinguisher of oscillations [16, 20] consists of a body with mass m_1 secured on an elastic support (case is equivalent to spring with rigidity K_1 connecting this body with fixed support), and

[1]State of an aircraft flying with subsonic speed is determined by magnitudes enumerated below and angle characterizing change of direction of air flow after flowing around the wing ("drift angle of flow") [18]. The latter depends on value of "angle of attack" of aircraft α (see below Fig. 5) at a moment of time, preceding that considered for certain finite quantity of τ. Therefore, the idea of a dynamic system does not fit subsonic aircraft. However, if one were to replace function $\alpha(t - \tau)$ by its approximation of the form $\alpha(t) - \tau\dot{\alpha}(t)$, as V. S. Vedrov does [13], then the situation may be corrected and the idea of a dynamic system may be spread to subsonic aircraft.

an extinguisher of oscillations constituting a body with mass $m_2 \ll m_1$ connected by a spring and damper (the latter not obligatory) with the first body (Fig. 4). An example of a mechanical system idealized as a body with mass m_1 elastically connected with fixed support can be a machine fixed on an elastic support. The dynamic extinguisher of oscillations is introduced for the decrease of amplitudes of oscillations of base, caused by unbalanced forces of variable directivity, appearing in the process of work of the machine. The considered system is a dynamic system if not one of the bodies is subjected to action of forces, besides the forces created by shown elastic couplings. In the mentioned example of a machine the considered system is dynamic if the machine is not working or if in the process of work of the machine, the base is not subjected to the influence of unbalanced forces.

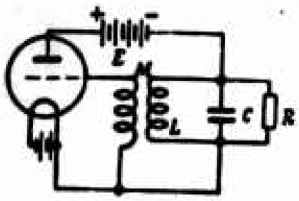

Fig. 3. Fundamental diagram of vaccuum-tube oscillator.

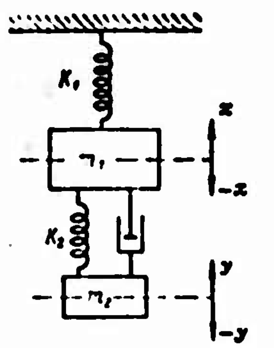

Fig. 4. Fundamental diagram of simplest elastic mechanical system with dynamic extinguisher of oscillations.

State of the considered system is determined by four magnitudes: displacements x and y of masses m_1 and m_2 and the speeds with which they shift, $\dot{x}$ and $\dot{y}$.

Basic characteristic of a dynamic system which is theoretically sufficient for studying processes occurring in it is a differential equation or a system of differential equations expressing its law of motion.

Since the state of a dynamic system at an arbitrary, fixed moment of time determines its motion in a subsequent interval of time, speed of change of magnitudes $x_1, \ldots, x_n$ at a considered moment of time have to be determined by values, at this instant of time, of shown magnitudes. Because of this, the law of motion of the system may be expressed by means of n first order differential equations [15],

$$\dot{x}_i = X_i(x_1, \ldots, x_n) \quad (i = 1, \ldots, n). \tag{1.1}$$

Thus, for instance, in the above-considered system consisting of a certain body, rod, and two springs (see Fig. 1), as variables determining state of system there were shown displacement x of center of gravity of body, relative to position of equilibrium, and speed of displacement $\dot{x}$. We will designate

$$x = x_1,$$
$$\dot{x} = x_2.$$

We will assume that in a certain bounded domain of change of variable x

$$-a<x<a \tag{1.2}$$

(where a is a positive number) forces of springs, effective on body, are proportional to displacement of its center of gravity x. Let us assume also that the system during motion experiences only dry friction, the force of which is constant in magnitude and is defined, with respect to direction, by sign of speed. Then condition of equilibrium of forces effective on body is reduced to equation

$$m\dot{x}_2+f\operatorname{sign}x_2+kx_1=0,$$

where f is magnitude of frictional force;

k is positive coefficient depending on elasticity of spring;

sign u is conditional designation of signum function:

$$\operatorname{sign}u\begin{cases}=1 & \text{when } u>0,\\ =0 & \text{when } u=0,\\ =-1 & \text{when } u<0.\end{cases}$$

If one were to supplement this equation by evident equality

$$\dot{x}_1=x_2,$$

then law of motion of system in region (1.2) will be expressed by system of two differential equations

$$\left.\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=-\frac{f}{m}\operatorname{sign}x_2-\frac{k}{m}x_1.\end{aligned}\right\} \tag{1.3}$$

In case of an electrical oscillation circuit (see Fig. 2) state of system, as was noted above, is determined by charge q and current intensity in circuit i. Equating potential difference on, connected in series, capacity and resistance of electromotive force of induction, we will obtain equation (21)

$$\frac{q}{C}+Ri=-L\frac{di}{dt}.$$

After designating

$$q=x_1,\ i=x_2$$

and taking into account that

$$i=\frac{dq}{dt},$$

we will obtain the following system of differential equations expressing the law of motion of the system:

$$\left.\begin{aligned}\dot{x}_1&=x_2,\\ \dot{x}_2&=-\frac{x_1}{CL}-\frac{Rx_2}{L}.\end{aligned}\right\}$$

Characteristic property of the law of motion of a dynamic system is its independence from external influences: motion of the system in a certain interval of time is fully determined by initial values of variables $x_1, \ldots, x_n$, and form of functions $X_i(x_1, \ldots, x_n)$ $(i = 1, \ldots, n)$ is determined by fundamental diagram and structural parameters of the system. In this meaning, it is possible to call a dynamic system "closed for external influences" or simply a "closed" system.

In nature and technology widespread also are systems not answering the definition of a dynamic system, but close to it in essence. Such systems are systems whose state at arbitrary fixed moment of time t, from certain interval of observation is fully (t_0, t_1), determined by values at this instant of time of n physical quantities presented by mathematical real variables $x_1, \ldots, x_n$, but motion of system in a subsequent interval of time is not determined by them.

As an example of such a system it is possible to consider supersonic aircraft flying in calm atmosphere with deflecting, in the process of flight, rudders, elevators and ailerons, if motion of the mentioned controls does not depend on motion of aircraft, for instance, occurs on a given program. In spite of the fact that the state of this system is determined by the same magnitudes which were indicated in examining the motion of an aircraft with secured rudders, elevators, and ailerons, the value of these magnitudes at any moment of time is not determined by subsequent motion of the system. In order to determine law of motion in this example, it is necessary to consider dependence of deflections of rudders, elevators and ailerons on time.

Law of motion of systems of considered form may be expressed by means of the following system of differential equations:

$$\dot{x}_i = X_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \qquad (i = 1, \ldots, n). \tag{1.4}$$

In this system variables $x_{n+1}, \ldots, x_{n+m}$ do not depend on process. In the considered example these variables were deflection of rudders, elevators, and ailerons.

Inasmuch as laws of change of variables $x_{n+1}, \ldots, x_{n+m}$ do not depend on process of motion of system, they can be represented as a function of time. If the last ones are known, then the law of motion may be determined in the form

$$\dot{x}_i = X_i(x_1, \ldots, x_n, t) \qquad (i = 1, \ldots, n). \tag{1.5}$$

Thus, for instance, motion of a supersonic aircraft with deflecting (independently of flight) rudder, elevators, and ailerons during symmetric flight with constant thrust conforms to the following system of several idealized differential equations:

$$m\dot{V} = -\frac{V^2}{2} S c_x(\alpha)\rho(H) - G\sin\theta + P\cos\alpha; \qquad mV\dot{\theta} = \frac{V^2}{2} S c_y(\alpha)\rho(H) - G\cos\theta + P\sin\alpha;$$

$$J_z\ddot{\vartheta} = m_z(\alpha, \delta)\frac{V^2 S b_A}{2}\rho(H) + m_z^{\bar{\omega}_z}\frac{V S b_A^2}{2}\rho(H)\dot{\vartheta} \qquad \dot{H} = V\sin\theta; \qquad \dot{L} = V\cos\theta. \tag{1.6}$$

Here V is speed of flight; ϑ is pitch angle (Fig. 5); θ is angle of inclination of trajectory; H is height of flight; L horizontal distance of flight; m is mass of aircraft; S is wing area; P is thrust; J_z is moment of inertia of aircraft with respect to an axis passing through its center of gravity of perpendicularly to plane of symmetry; b_A is average aerodynamic chord; $\alpha = \vartheta - \theta$ is angle of attack; δ is elevator angle, c_x, c_y, m_z, $m_z^{\bar{\omega}_z}$ is aerodynamic coefficients which are functions of magnitudes shown in parentheses; ρ is air density (function of height of flight).

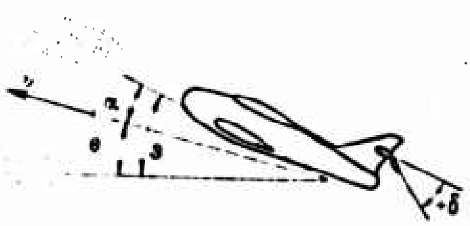

Fig. 5. Kinematic parameters of aircraft.

Variables determining state of system here are: speed and height of flight V and H, angles of pitch and trajectory ϑ and θ, speed of change of pitch $\dot{\vartheta}$, horizontal flying range L. A variable not depending on process is elevator angle δ.

Designating

$$V = x_1,\ H = x_2,\ \theta = x_3,\ \vartheta = x_4,\ \dot{\vartheta} = x_5,\ L = x_6,\ \delta = x_7,$$

it is possible to record equation (1.6) in the form

$$\left.\begin{aligned}
\dot{x}_1 &= X_1(x_1, x_2, x_3, x_4),\\
\dot{x}_2 &= X_2(x_1, x_4),\\
\dot{x}_3 &= x_5,\\
\dot{x}_4 &= X_4(x_1, x_2, x_3, x_4),\\
\dot{x}_5 &= X_5(x_1, x_2, x_3, x_4, x_5, x_7),\\
\dot{x}_6 &= X_6(x_1, x_4).
\end{aligned}\right\} \tag{1.7}$$

If elevator is deflected according to a known program, then, after replacing variable x_7 with function of time, there can be obtained a system of the form (1.5).

Characteristic property of laws of motion (1.4) and (1.5) is their dependence on external influences. In the case of law of motion (1.4), a source of external influences change of variables x_{n+1}, ..., x_{n+m}. In the case of law of motion (1.5), an effect of external influences is hidden in the functional dependences of magnitudes (i = 1, ..., n) on time t.

Motion of systems of the considered new form in a certain interval of time is determined by initial values of variables x_1, ..., x_n and external influences. In this meaning, systems of given form it is possible to call "open for external influences"

or simply "open" systems. We will call them <u>dynamic</u> just as were called earlier-considered systems, and subsequently, when speaking of dynamic systems, we will keep in mind that these may be both closed and open dynamic systems.

In the example of a supersonic aircraft it was shown how a closed dynamic system, by means of increasing number of moving elements (releasing rudder, elevators and ailerons) can be turned into an open system. On the same example it will show how, during further increase in number of moving elements, from this system we can again make a closed dynamic system.

In the considered example it was mentioned that motion of elevator does not depend on change of other variables characterizing motion of system. We will reject this condition and will connect motion of control surface with change in pitch angle of aircraft with the help of the automatic pilot. Let us assume that an equation of the latter, with a certain degree of idealization, has the form

$$\dot{\delta} + T\delta = i_0\vartheta + i_1\dot{\vartheta},$$

where T is time constant of automatic pilot; i_0, i_1 its transmission ratio.

Assuming that magnitudes T, i_0 and i_1 are constants, it is possible to record this equation (in earlier taken designations) in the form

$$\dot{x}_7 = X_7(x_3, x_5, x_7). \qquad (1.8)$$

Supplementing, with this equation, system (1.7), there can be obtained a system of equations expressing the law of motion of a closed dynamic system.

The dynamic system, aircraft — automatic pilot, in the considered form is indeed closed. Its state is determined by seven magnitudes, x_1, ..., x_7.

Above it was indicated that for closed dynamic systems there is a characteristic independence of laws of their motion on external influences, i.e., a certain "autonomy" of processes occurring in them. Therefore, closed dynamic systems frequently are called <u>autonomous</u> and, correspondingly, open dynamic systems — <u>nonautonomous</u> systems.

The automatic pilot in the considered example is an open dyanmic system. In this case, the closed dynamic system consists of two open systems with laws of motion expressed by equations

$$\dot{x}_i = X_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \quad (i = 1, \ldots, n)$$

and

$$\dot{x}_i = X_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \quad (i = n+1, \ldots, n+m),$$

which, in totality, express laws of motion of systems on the whole.

Systems of this form, at present, have obtained very wide propagation in different

egions of technology as systems of automatic control.

§ 2. Linearity of Large and Small Systems

As was shown above, the law of motion of an open dynamic system may be expressed by system of equations (1.4). Since system of equations (1.1), expressing law of motion of a closed dynamic system, it is possible to consider as a particular case of this system, subsequently we will consider that system (1.4) expresses law of motion of a dynamic system of general form.

If functions $X_i(x_1, \ldots, x_n, t)$ $(i = 1, \ldots, n)$ in equations (1.5) in a certain region of variation of variables $x_1, \ldots, x_{n+m}$, t

$$\left.\begin{array}{l} a_i' < x_i < a_i'', \\ t_0 < t < t_1, \end{array}\right\} \qquad (1.9)$$

$(i = 1, \ldots, n + m,\ t_1 \leq \infty,\ t_1 - t_0 \geq b$, where b is final positive value) are linear relative to variables $x_1, \ldots, x_n$), then for any given set of functions $x_{n+1}(t), \ldots, x_{n+m}(t)$ equations (1.4) in region (1.9) it is possible to present in the form

$$\dot{x}_i = a_{i1}(t)x_1 + \ldots + a_{in}(t)x_n + Y_i(t) \qquad (i = 1, \ldots, n), \qquad (1.10)$$

where $a_{ij}(t)$ and $Y_i(t)$ $(i, j = 1, \ldots, n)$ are real functions from t.

Dynamic systems, whose law of motion in region of variation of variables (1.9) is expressed by equations (1.10), are called linear in this region.

For real linear dynamic systems, the region of variation of variables, in which the system is linear, must be limited. It may be either finite or infinitesimal.

If region (1.9) is finite, i.e., if $a_i'' - a_i'$ $(i = 1, \ldots, n + m)$ are finite numbers, then system is called linear large.

If in region (1.9) all or certain intervals (a_i', a_i'') $(i = 1, \ldots, n + m)$ are infinitesimal, then system is called linear small.

The idea of linearity large, in view of its clarity, does not require further explanation. We will explain somewhat in greater detail the idea of linearity small.

Let us assume that there is known a certain particular solution of system (1.4)

$$x_i = x_i^{(0)}(t) \qquad (i = 1, \ldots, n). \qquad (1.11)$$

which in particular cases may be, for instance, solution

$$x_i^{(0)}(t) = C_i \qquad (i = 1, \ldots, n)$$

[where C_i ($i = 1, \ldots, n$) are real constants] of system of equations

$$\dot{x}_i = X_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = 0$$
$$(i = 1, \ldots, n),$$

expressing interrelation between variables $x_1, \ldots, x_{n+m}$ in a state of rest (see § 3).

Considering solution $x_i(t)$ ($i = 1, \ldots, n$) not coinciding with solutions $x_i^{(0)}(t)$, we will introduce variables $x_i'(t)$ by formulas [15]:

$$x_i'(t) = x_i(t) - x_i^{(0)}(t) \quad (i = 1, \ldots, n+m). \tag{1.12}$$

Carrying out substitution (1.12) in system (1.4), we will obtain

$$\dot{x}_i' = X_i'(x_1', \ldots, x_n', x_{n+1}', \ldots, x_{n+m}') \quad (i = 1, \ldots, n), \tag{1.13}$$

where $X_i'(x_1', \ldots, x_n', x_{n+1}', \ldots, x_{n+m}')$ are certain new functions.

We will assume that functions $X_i'(x_1', \ldots, x_{n+m}')$ ($i = 1, \ldots, n$) in a certain region of variation of magnitudes $x_i'(t)$

$$-A_i < x_i'(t) < A_i \quad (i = 1, \ldots, n+m, \; t_0 \leqslant t < t_1) \tag{1.14}$$

[where A_i ($i = 1, \ldots, n + m$) are finite numbers] are expandable into an absolutely converging Taylor series in degrees of these magnitudes [6].

Assuming that considered solution $x_i(t)$ ($i = 1, \ldots, n$) is included in region (1.14) and carrying out mentioned expansion, we will obtain from system (1.13) system

$$\dot{x}_i = a_{i1}(t)x_1' + \ldots + a_{in}(t)x_n' + a_{i,n+1}(t)x_{n+1}' + \ldots + a_{i,n+m}(t)x_{n+m}' + \text{remainder} \quad (i = 1, \ldots, n). \tag{1.15}$$

With this

$$a_{ij}(t) = \left(\frac{\partial X_i'}{\partial x_j'}\right)_{x_1' = \ldots = x_{n+m}' = 0}, \tag{1.16}$$

and remainder contains second and highest degrees of magnitudes $x_1', \ldots, x_{n+m}'$.

In any finite interval (t_0, t_1) remainders approach if

$$\left.\begin{aligned} & x_i'(t_0) \to 0 \quad (i = 1, \ldots, n) \\ \text{and} \quad & A_i \to 0 \; (i = n+1, \ldots, n+m). \end{aligned}\right\} \tag{1.17}$$

this gives cause to affirm that, while considering oscillation in a finite interval,

during numerically rather small magnitudes $x_i'(t_0)$ $(i = 1, \ldots, n)$ and A_i $(i = n + 1, \ldots, n + m)$ it is possible to disregard them.

Such affirmation is indeed true: however, the correctness of it is not evident. The fact is that during condition (1.17), along with the remainders, also approaching zero are all other magnitudes in the right sides of the equations (1.15). Consequently, on the basis of this property, we cannot say that solution of systems of equations (1.15) with remainders and without them are in some meaning close.

In order to clarify the possiblility and meaning of disregarding remainders in equations (1.15) during condition (1.17), we will present variables $x_1', \ldots x_n'$ in the form

$$x_i' = re_i \qquad (i = 1, \ldots, n), \tag{1.18}$$

where $r = \sqrt{(x_1')^2 + \ldots + (x_n')^2}$, e_i $(i = 1, \ldots, n)$ are magnitudes numerically not exceeding unity.

Carrying out substitution (1.18) into equations (1.15) and dividing the latter term by term by r, we will obtain

$$\left.\begin{aligned} \dot{e}_1 &= \left(a_{11} - \frac{\dot{r}}{r}\right)e_1 + a_{12}e_2 + \ldots + a_{1n}e_n + \frac{a_{1,n+1}x_{n+1}'}{r} + \ldots \\ &\quad \ldots + \frac{a_{1,n+m}x_{n+m}'}{r} + \text{remainder} \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{e}_n &= a_{n1}e_1 + a_{n2}e_2 + \ldots + \left(a_{nn} - \frac{\dot{r}}{r}\right)e_n + \frac{a_{n,n+1}x_{n+1}'}{r} + \ldots + \frac{a_{n,n+m}x_{n+m}}{r} + \text{remainder} \end{aligned}\right\} \tag{1.19}$$

We will estimate remainders of these equations. Let us consider, for concreteness, remainder of first equation.

In accordance with substitution (1.18), remainder of i-th equation of system (1.15), divided by r. Therefore, instead of remainders of equations of system (1.19), it is possible to consider remainders of equations of system (1.15).

Remainder of first equation of system (1.15), according to the rule of expansion of a function of several variables in Taylor series, has the form

$$\sum_{l=2}^{\infty} \sum_{\substack{i_1, \ldots, i_{n+m}=0 \\ (i_1+\ldots+i_{n+m}=l)}}^{l} P(i_1, i_2, \ldots, i_{n+m})(x_1')^{i_1}(x_2')^{i_2} \ldots (x_{n+m}')^{i_{n+m}},$$

where coefficients $P(i_1, i_2, \ldots, i_{n+m})$ are determined for every value of t, are real and limited.

Let us assume that M(t) is a certain upper bound of modulus of difference

$$X_1'(x_1', \ldots x_{n+m}', t) - a_{11}x_1' - \ldots - a_{1,n+m}x_{n+m}'$$

during all possible values of magnitudes $x_1', \ldots, x_{n+m}'$, moduli of which, respectively,

are equal to A_1, ..., A_{n+m}. Then, by known theorem of analysis, we will obtain

$$|P(i_1, i_2, \ldots, i_{n+m})| A_1^{i_1} \ldots A_{n+m}^{i_{n+m}} < M(t)$$

and as a result

$$|P(i_1, i_2, \ldots, i_{n+m})(x_1')^{i_1} \ldots (x_{n+m}')^{i_{n+m}}| < M(t). \tag{1.20}$$

Numbers of members

$$P(i_1, i_2, \ldots, i_{n+m})(x_1')^{i_1} \ldots (x_{n+m}')^{i_{n+m}}$$

with the same value of sum

$$i_1 + i_2 + \ldots + i_{n+m} = i$$

does not exceed number $(n + m)^i$. Therefore, because of inequality (1.20), equalities (1.18), and condition $|e_i| \leq 1$, modulus of sum interesting us does not exceed magnitude

$$M(t) \sum_{i=2}^{\infty} (n+m)^i \{\max [r(t), A_{n+1}, \ldots, A_{n+m}]\}^i.$$

Assuming that $(n + m)\{\max[r(t), A_{n+1}, \ldots, A_{n+m}]\} < 1$, after designating

$$\max(r, A_{n+1}, \ldots, A_{n+m}) = N$$

and applying formula of sum of geometric progression, we will find

$$\sum_{i=2}^{\infty} (n+m)^i [\max(r, A_{n+1}, \ldots, A_{n+m})]^i = \frac{(n+m)^2 N^2}{1-(n+m)N}.$$

Consequently, modulus of estimated sum during expressed-above assumption does not exceed magnitude

$$\frac{MN^2(n+m)^2}{1-(n+m)N}.$$

Dividing this expression by r, we will obtain appraisal for modulus of remainder of first equation of system (1.19).

Analogous appraisals, obviously, can be obtained for remainders of other equations of systems (1.19) and (1.15).

Now it is possible to compare solution of system (1.15) with solution of system obtained from it, as a result of disregarding remainders. It is natural to assume the following criterion of proximity of mentioned solutions: comparable solutions are as close as possible during condition (1.17) only when for any (as small as desired) positive number ε there can be designated an initial value of magnitude r and values of magnitudes A_{n+1}, ..., A_{n+m}, so small that difference of analogous magnitudes

$$\frac{x_i'}{r} = e_i$$

in these solutions for all i will not numerically exceed ε.

Because of formulated criterion, proximity of comparable solutions is determined by proximity of corresponding solutions of system (1.19) and a system obtained from

the latter as a result of disregarding remainders. If one were to consider that magnitude $\dot{r}/r$ during sufficiently small initial values of r is limited in any finite interval, then it is possible to show that during condition (1.17) the mentioned solutions of system (1.19) and the simplified system corresponding to it are as close as possible. From this follows the possibility of disregarding remainders of equations (1.15) in examining oscillations in a finite interval on the assumption that initial values $x_i'(t_0)$ $(i = 1, \ldots, n)$ and intervals $(-A_i, A_i)$ $(i = n + 1, \ldots, n + m)$, in which are included variables $x_{n+1}', \ldots, x_{n+m}'$ are sufficiently small.

Such a possibility, in general, is lost during transition to open interval of time. Exceptions are cases when during certain finite values of magnitudes A, $A_{n+1}, \ldots, A_{n+m}$, for all solutions satisfying conditions

$$\begin{gathered} 0 < r(t_0) < A, \\ -A_i < x_i'(t) < A_i \\ (i = n+1, \ldots, n+m), \end{gathered}$$

during $t \to \infty$, magnitude $r(t)/r(t_0)$ has finite upper limit. Oscillations at which this condition is executed are stable according to Lyapunov (see § 1, Ch. V) with respect to magnitudes $x_1, \ldots, x_n$, if $x_{n+1}' = \ldots = x_{n+m}' = 0$, and at $x_1'(t_0) = \ldots = x_n'(t_0) = 0$ they are limited in change of variables $x_1', \ldots, x_n'$ in such a way that the last ones do not emerge beyond boundaries of interval $(-\varepsilon, \varepsilon)$ as small as desired during sufficiently small values of magnitudes $A_{n+1}, \ldots, A_{n+m}$. Considering both these properties, the oscillations satisfying the above-mentioned condition it is possible to call <u>double stable with respect to magnitudes</u> $x_1, \ldots, x_n$. It is necessary to note that in the case of closed dynamic systems, because of absence of variables $x_{n+1}, \ldots, x_{n+m}$, the idea of double stable oscillations loses meaning and only stability according to Lyapunov can be discussed.

From that presented it follows that: a) <u>a dynamic system is linear small during initial values of magnitudes</u> $x_1, \ldots, x_n$, <u>close to values</u> $x_1^{(0)}(t_0), \ldots, x_n^{(0)}(t_0)$, <u>and small deviations of variables</u> $x_{n+1}, \ldots, x_{n+m}$ <u>from variables</u> $x_{n+1}^{(0)}, \ldots, x_{n+m}^{(0)}$ <u>in arbitrary finite interval</u> (t_0, t_1), <u>if functions</u> $X_i'(x_1', \ldots, x_{n+m}')$ $(i = 1, \ldots, n)$, <u>are expandable in this interval, in degrees of magnitudes</u> $x_1', \ldots, x_n$, <u>into an absolutely converging Taylor series;</u> b) <u>a dynamic system is linear small with those same conditions of proximity of variables</u> x_i and $x_i^{(0)}$ $(i = 1, \ldots, n + m)$ <u>in open interval</u> (t_0, ∞), <u>if functions</u> $X_i'(x_1', \ldots, x_{n+m}')$ <u>are expandable in this interval into an absolutely converging Taylor series and oscillation presented by solution of system</u>

(1.15)

$$x_i^{(0)}(t) \quad (i=1, \ldots, n)$$

are double stable in the above-indicated meaning with respect to magnitudes x_1, ..., x_n).

If there is given system of function $x'_{n+1}(t)$, ..., $x'_{n+m}(t)$, the values of which in the considered interval of change of variable t are sufficiently small, then after disregarding remainders, the system of equation (1.15) can be rewritten in the form

$$\dot{x}'_i = a_{i1}x'_1 + \ldots + a_{in}x'_n + Y'_i(t) \quad (i=1, \ldots, n). \tag{1.21}$$

Dropping primes for variables x_i and Y_i ($i = 1, \ldots, n$), we will obtain system of equations (1.10). Allowing this freedom in designations, the law of motion of dynamic systems, both linear large and linear small, it is possible to express by a system of differential equations (1.10). With this, however, one should not forget about the difference in physical meaning of variables x_i and Y_i ($i = 1, \ldots, n$), in these cases.

Subsequently, speaking of linear systems, we will not discuss what kind of regions are the regions variation of variables $x_1, \ldots, x_{n+m}$ in which they are linear, but we will assume that values of variables $x_1, \ldots, x_{n+m}$ in oscillations considered by us do not exceed the bounds of these regions. In connection with this, considering oscillations of linear systems <u>unstable</u> we will consider that unlimited growth of moduli of variables $x_1, \ldots, x_n$ does not have a physical meaning and that the idea of instability one should physically consider only so that the moment of time will set in, which may be any amount distant, when any of the variables $x_1, \ldots, x_n$ will go beyond the boundaries of the region of linearity.

<u>Example 1.</u> Mechanical system, consisting of body with mass m, rod, and two springs (see Fig. 1), given as the simplest example of the dynamic system in the preceding paragraph, has, under the assumptions made there, a law of motion expressed by the following system of differential equations,

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{k}{m}x_1 - \frac{f}{m}\operatorname{sign} x_2. \end{aligned}\right\}$$

(see § 1), where x_1 is displacement of center of gravity of body relative to position of equilibrium. Region of variation of variables, for which this system of differential equations was limited by one condition (1.2)

$$-a < x < a,$$

where a is a positive number.

Obviously, the considered dynamic system is not linear large, since not even in one bounded domain of change of variables x_1 and x_2, is the right side of the second equation a linear function of variable x_2. This dynamic system is also not linear small since function

$$X_2(x_1, x_2) = -\frac{k}{m} x_1 - \frac{f}{m} \operatorname{sign} x_2,$$

not being differentiable with respect to x_2, cannot be even in one bounded domain of change of arguments expanded into Taylor series.

Thus, this dynamic system is <u>nonlinear.</u>

<u>Example 2.</u> Electrical oscillation circuit, depicted on Fig. 2, has the law of motion expressed by system of differential equations

$$\left.\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{1}{LC} x_1 - \frac{R}{L} x_2. \end{aligned}\right\}$$

where $x_1 = q$ is charge; $x_2 = i$ is current intensity.

If parameters of circuit R, L, and C are constant in a certain bounded domain of change of charge q and current intensity i, then this dynamic system is a <u>system linear large.</u>

<u>Example 3.</u> Supersonic aircraft with deflecting (independently of flight) elevator during symmetric flight in assumptions expressed in the preceding paragraph, in a particular case, can fly rectilinearly, horizontally with constant speed. In this case $\theta = \dot{\vartheta} = V = 0$. Values of variables ϑ, V, H and function L(t) are defined as solutions of system of equations

$$\left.\begin{aligned} -\frac{V^2 S c_x(\vartheta)\rho(H)}{2} + P\cos\vartheta &= 0, \\ \frac{V^2 S c_y(\vartheta)\rho(H)}{2} - G + P\sin\vartheta &= 0, \\ m_z(\vartheta, \delta) - V^2 S b_A \rho(H) &= 0, \\ \dot{L} &= V, \end{aligned}\right|$$

into which is transformed system (1.6) for the mentioned case of flight, and they depend on assumed value of elevator deflection δ. Designating these values of variables and junction L(t) by symbols $x_1^{(0)}$, $x_2^{(0)}$, $x_3^{(0)}$, $x_6^{(0)}$, $x_7^{(0)}$ in the order shown in § 1, assuming that functions $c_x(\vartheta)$, $c_y(\vartheta)$, $m_z(\vartheta, \delta)$, $\rho(H)$ are expanded into converging Taylor series in a certain bounded domain of change of variables ϑ, δ, H, found values of which are its center, and considering that $x_4^{(0)} = 0$, we will obtain from system (1.7) a system of linear differential equations with constant

coefficients relative to $x_i' = x_i - x_i^{(0)}$ $(i = 1, \ldots, 7)$. This system of equations will express law of motion of the considered dynamic system which is, thus, a system linear small.

§ 3. Forced and Free Oscillations of Linear Systems

We will define state of rest of a linear dynamic system, whose law of motion is expressed by system of equations (1.10), by condition

$$\dot{x}_i = 0 \qquad (i = 1, \ldots, n). \tag{1.22}$$

From this condition, because of mentioned system of equations it follows that

$$a_{i1}(t)x_1 + \ldots + a_{in}(t)x_n = -Y_i(t)(i = 1, \ldots, n). \tag{1.23}$$

Let us assume that under certain influences $Y_i(t)$ $(i = 1, \ldots, n)$ at initial moment of time $t = t_0$ the system was in a state of rest. Let us assume further that determinant of system (1.23)

$$\begin{vmatrix} a_{11}(t), & \ldots, & a_{1n}(t) \\ \ldots & \ldots & \ldots \\ a_{n1}(t), & \ldots, & a_{nn}(t) \end{vmatrix}$$

does not turn into zeroes at $t \geq t_0$ or at least at $t = t_0$. Then values of influences Y_i $(i = 1, \ldots, n)$ at initial moment of time simply determine values of magnitudes $x_1, \ldots, x_n$ at this instant of time. After determining value of these magnitudes and placing them in equations (1.23), it is possible to determine simply that form of influences $Y_i(t)$ at which system remains in state of rest in the subsequent, considered interval of time. We will designate these influences by symbols $Y_i^{(0)}(t)$ $(i = 1, \ldots, n)$. From determination, obviously,

$$Y_i(t_0) = Y_i^{(0)}(t_0) \qquad (i = 1, \ldots, n). \tag{1.24}$$

Let us assume that $x_i^{(0)}$ $(i = 1, \ldots, n)$ is value of magnitudes x_i in considered state of rest. We will introduce new variables Δx_i and ΔY_i:

$$\left.\begin{aligned} \Delta x_i &= x_i(t) - x_i^{(0)}, \\ \Delta Y_i &= Y_i(t) - Y_i^{(0)}(t) \\ & (i = 1, \ldots, n). \end{aligned}\right\} \tag{1.25}$$

Then system of equations (1.10) in new variables will take the form

$$\Delta\dot{x}_i = a_{i1}(t)\Delta x_1 + \ldots + a_{in}(t)\Delta x_n + \Delta Y_i(t) \\ (i = 1, \ldots, n). \tag{1.26}$$

With this, according to determination,

$$\Delta x_i(t_0) = \Delta Y_i(t_0) = 0 \qquad (i = 1, \ldots, n). \tag{1.27}$$

If in considered interval of time

$$Y_i(t) = Y_i^{(0)}(t), \tag{1.28}$$

then also, according to definition, the state of rest of the system in this interval is not disturbed. If, however, there exist intervals of time belonging to the considered interval, when equalities (1.28) do not hold, then the system comes to a state of motion.

This motion is called forced oscillations.

Now, preserving the introduced designations, we will assume that in the initial moment of time the system was not in a state of rest, determined by initial values of influences $Y_i(t)$ ($i = 1, \ldots, n$), but form of influences is such that they satisfy condition $Y_i = Y_i^0$. Then system will come into a state of motion which is called free oscillations.

System of equations (1.10), in this case, in new variables takes the form

$$\Delta \dot{x}_i = a_{i1}(t)\Delta x_1 + \ldots + a_{in}(t)\Delta x_n \quad (i = 1, \ldots, n). \tag{1.29}$$

Cases of forced and free oscillations are particular cases of motion. In the general case of motion, there simultaneously exist causes provoking forced and free oscillations. Because of linearity of system, its motion in this case constitutes the sum of forced and free oscillations determined independently from each other.

§ 4. Equation of Free Oscillations

Usually during the study of free oscillations of a linear system, not all variables $x_1, x_2, \ldots, x_n$ present interest. In practice, very frequently we are interested in the behavior of only one variable, more rarely of two variables, and very rarely of a large number of variables.

If the subject of study is the behavior of variable x_1, then during certain conditions a system of differential equations (1.29) may be reduced to one differential equation (0.1) of n-th order

$$\frac{d^n x}{dt^n} + b_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + b_n(t)x = 0,$$

in which coefficients $b_1(t), \ldots, b_n(t)$ are functions of coefficients $a_{ij}(t)$ ($i, j = 1, \ldots, n$) and their derivatives. This conversion may be carried out in the following manner.

Differentiating first equation of system (1.29), we will obtain

$$\Delta\ddot{x}_1 = \dot{a}_{11}\Delta x_1 + \ldots + \dot{a}_{1n}\Delta x_n + a_{11}\Delta\dot{x}_1 + \ldots + a_{1n}\Delta\dot{x}_n.$$

Placing instead of derivatives $\Delta\dot{x}_1, \ldots, \Delta\dot{x}_n$ right sides of corresponding equations of system (1.29), we will find

$$\Delta\ddot{x}_1 = \left(\dot{a}_{11} + \sum_{i=1}^{n} a_{1i}a_{i1}\right)\Delta x_1 + \ldots + \left(\dot{a}_{1n} + \sum_{i=1}^{n} a_{1i}a_{in}\right)\Delta x_n.$$

Differentiating this equation and, liberating it, analogously, from derivatives $\Delta\dot{x}_1, \ldots, \Delta\dot{x}_n$, we will obtain an equation expressing third derivative of magnitude Δx_1 through magnitudes $\Delta x_1, \ldots, \Delta x_n$, etc.

Carrying out such an operation n-1 times, we will obtain a system of equations of the form

$$\left.\begin{array}{l} \dfrac{d\Delta x_1}{dt} = b_{11}\Delta x_1 + \ldots + b_{1n}\Delta x_n, \\ \cdots\cdots\cdots\cdots\cdots \\ \dfrac{d^n\Delta x_1}{d(\Delta x_1)^n} = b_{n1}\Delta x_1 + \ldots + b_{nn}\Delta x_n. \end{array}\right\} \qquad (1.30)$$

in which coefficients b_{1i} $(i = 1, \ldots, n)$ are polynomials from coefficients a_{ij} $(i, j = 1, \ldots, n)$ and first derivative coefficients $a_{11}, \ldots, a_{1n}$; coefficients b_{2i} $(i = 1, \ldots, n)$ are polynomials from coefficients a_{ij} $(i, j = 1, \ldots, n)$, their first derivatives and second derivatives of coefficients $a_{11}, \ldots, a_{1n}$; ... coefficients b_{ni} $(i = 1, \ldots, n)$ are polynomials from coefficients a_{ij} $(i, j = 1, \ldots, n)$, their derivatives, from first to n-2nd, inclusively, and n-1st derivatives of coefficients $a_{11}, \ldots, a_{1n}$.

If determinant of system (1.30), considered as algebraic system of linear equations relative to unknowns $\Delta x_1, \ldots, \Delta x_n$, at all values of t from interval interesting us, is different than zero, then we will assume that solving this system relative to Δx_1 and considering $\Delta x_1 = x$, we will obtain equation

$$\frac{B_n}{\Delta}\frac{d^n x}{dt^n} + \frac{B_{n-1}}{\Delta}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + \frac{B_1}{\Delta}\frac{dx}{dt} - x = 0,$$

in which Δ is determinant of system (1.30),

$$B_i = (-1)^{i+1}\begin{vmatrix} b_{12}, \ldots, b_{1n} \\ \cdots\cdots\cdots \\ b_{i-1,2}, \ldots, b_{i-1,n} \\ b_{i+1,2}, \ldots, b_{i+1,n} \\ \cdots\cdots\cdots \\ b_{n2}, \ldots, b_{nn} \end{vmatrix}$$

$$(i = 1, \ldots, n).$$

After assuming additionally that $B_n \neq 0$ for all values of t from considered interval and multiplying, term by term, the latter equation by Δ/B_n, we will obtain an

equation of the form (0.1), in which

$$b_i = \frac{B_{n-i}}{B_n} \quad (i = 1, \ldots, n-1)$$

$$b_n = -\frac{\Delta}{B_n}.$$

Thus, for instance, system of the second order is reduced to equation

$$\ddot{x} + b_1\dot{x} + b_2 x = 0,$$

in which

$$b_1 = -\frac{\dot{a}_{12}}{a_{12}} - a_{11} - a_{22},$$

$$b_2 = \frac{a_{11}\dot{a}_{12}}{a_{12}} - \dot{a}_{11} + a_{11}a_{22} - a_{12}a_{21}.$$

From that presented, it follows that conversion of system (1.29) into equation (0.1) is possible if

a) certain relationships are satisfied between coefficients a_{ij} (i, j - 1, ..., n), at which $B_n \neq 0$ and $\Delta \neq 0$;

b) coefficients a_{1i} (i = 1, ..., n) n-1 multiple are differentiable;

c) coefficients a_{ji} (i = 1, ..., n; j = 2, ..., n) n-2 multiple are differentiable.

We will consider that there are considered only such laws of motion at which these conditions are carried out. Moreover, subsequently, we will assume that coefficients $b_1(t)$, ..., $b_n(t)$ during all values of t from the considered interval, which may be either interval (0, ∞) or a part of it, are continuous and are differentiable, where they are limited at $t = t_0$.

Equation (0.1) we will call equation of free oscillations.

§ 5. Linear Systems with Variable Parameters

If the dynamic system is linear in large and variables x_1, ..., x_n, in process of motion, do not exceed the bounds of the region of linearity, then coefficients b_1, ..., b_n of the equation of free oscillations are determined by construction of the system and law of change of magnitudes x_{n+1}, ..., x_{n+m}. If the system is linear in small, then these coefficients are determined by construction of system and dependencies $x_1^{(0)}(t)$, ..., $x_{n+m}^{(0)}(t)$, characterizing change of variables x_1, ..., x_{n+m} in the process of that motion, small oscillations near which are investigated. We will call coefficients $b_1(t)$, ..., $b_n(t)$, following accpeted terminology, parameters of the system, remembering, however, their real physical meaning.

If all coefficients of the equation of free oscillations are constant, then the linear system with respect to variable x is a linear system with constant parameters.

If in the equation of free oscillations at least one of coefficients is inconstant, then the linear system with respect to variable x is called a linear system with variable parameters.

Parameters $b_1(t)$, ..., $b_n(t)$ of natural and technical linear systems with variable parameters can have the most diverse form. If the form of parameters is put in the basis of classification of linear systems with variable parameters, then from the general class of systems it is possible to separate two subclasses, representatives of which are encountered very frequently in nature and technology. First subclass is formed by linear systems with periodically variable parameters, i.e., such linear systems for which coefficients $b_1(t)$, ..., $b_n(t)$ are periodic functions of time of one and the same period Ω (certain coefficients, in particular, can be constants). The second subclass is formed by linear systems for which coefficients $b_1(t)$, ..., $b_n(t)$ of the equation of free oscillations are power or composite power functions of time, i.e., have the form

$$b_i = B_i t^{\beta_i},$$

or

$$b_i = \frac{\sum_{j=1}^{l} B_{ij} t^{\beta_{ij}}}{\sum_{j=1}^{m} C_{ij} t^{\gamma_{ij}}},$$

where B_i, B_{ij}, C_{ij}, β_i, β_{ij}, γ_{ij} for all possible values of indices i and j are constants.

Theory of oscillations of linear systems with variable parameters of general form is expounded in the following four chapters. More detailed research of oscillations of the systems of above-indicated special form are presented in the last two chapters.

§ 6. Certain Ideas and Determinations from a Theory of Differential Equations

Let us assume that t_0 is any value of t, belonging to considered interval, and ξ_0, ξ_1, ..., ξ_{n-1} is a given system of complex numbers. Then, there exists one and only one particular solution x(t), satisfying in this interval equation ([illegible]), for which $t = t_0$ derivative of i-st order ($i = 0, 1, 2, 3, \ldots, n-1$) turns into ξ_i.

If functions

$$x^{(1)}(t),\ x^{(2)}(t), \ldots,\ x^{(m)}(t) \qquad (1.[illegible])$$

are particular solutions of equation (0.1), then any linear combination of them with constant coefficients

$$\sum_{k=1}^{m} C_k x^{(k)}(t)$$

also is its particular solution.

Solution (1.31) is called linearly independent if such constnat C_1, C_2, C_3, C_4, ..., C_m do not exist, among which at least one is different than zero, for which occurs identity

$$\sum_{k=1}^{m} C_k x^k(t) = 0; \tag{1.32}$$

If this condition is not executed, solutions are called <u>linearly dependent</u>. There exists a system of n linearly independent particular solutions of equation (0.1). Such a system of solutions is called <u>fundamental</u> [23].

Function x(t), depending on n arbitrary constant C_1, C_2, ..., C_n, satisfying equation (0.1) and transformed at corresponding values of these constants into any particular solution, is called the <u>general solution</u> of this equation [23].

If particular solutions (1.31) form fundamental system of solutions, then m = n and general solution is given by formula

$$x(t) = C_1 x^{(1)}(t) + C_2 x^{(2)}(t) + \ldots + C_n x^{(n)}(t). \tag{1.33}$$

In the process of studying properties of equation (0.1), in the following chapters will appear linear differential equations of the form (0.1) with complex coefficient $b_n(t)$, which is a continuous and differentiable function of argument t, and systems of first-order differential equations, solved relative to the derivative

$$\dot{x}_i = f_i(t, x_1, x_2, \ldots, x_n) \quad (i = 1, 2, 3, \ldots, n). \tag{1.34}$$

The right sides of these derivatives are continuous, complex-valued functions of argument t and unknown variables, definite in a certain region G of space (t, x_1, x_2, ..., x_n).

Complexity of coefficient $b_n(t)$ of equation (0.1), as any other coefficients of this equation, does not change above-mentioned ideas of theory of linear uniform differential equation; all above-indicated formulations are valid also in this case [2?].

<u>Particular solution of system of differential equations (1.34)</u> is a system of single-valued functions

$$x_1(t), x_2(t), \ldots, x_n(t).$$

satisfying (during joint substitution) these equations.

Set of functions

$$x_i = x_i(t_1, C_1, C_2, \ldots, C_m) \quad (i = 1, 2, 3, \ldots, n), \tag{1.35}$$

depending on arbitrary constants C_1, C_2, ..., C_m, is called general solution of system (1.5) in region G, if selecting in the appropriate way constants C_1, C_2, ... C_m, there can be obtained any particular solution occurring in this region.

Particular case of system (1.34) is a system of linear uniform differential equations

$$\dot{x}_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \quad (i = 1, 2, 3, \ldots, n), \tag{1.36}$$

coefficients of which are continuous, but not necessarily real.

Linear combination of particular solutions

$$x_1^{(k)}(t), \ldots, x_n^{(k)}(t) \qquad (k = 1, 2, 3 \ldots, m)$$

of system (1.36) is set of functions

$$\sum_{k=1}^{m} C_k x_1^{(k)}(t), \quad \sum_{k=1}^{m} C_k x_2^{(k)}(t), \ldots, \quad \sum_{k=1}^{m} C_k x_n^{(k)}(t),$$

where C_1, C_2, C_3, ..., C_m are certain constants.

Linear combination of particular solutions of system (1.36) also is its particular solution. Particular solutions

$$x_i^{(1)}(t) \qquad (i = 1, 2, \ldots, n),$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$x_i^{(m)}(t) \qquad (i = 1, 2, \ldots, n)$$

are called linearly dependent on each other if there exist such constants C_1, C_2, ..., C_m, among which there is at least one different than zero, which, during any t, take the place of identity

$$\sum_{k=1}^{m} C_k x_i^{(k)}(t) = 0 \, (i = 1, 2, \ldots, n);$$

If shown conditions do not hold, particular solutions are called linearly independent.

System of n linearly independent particular solutions of system (1.36) is called its fundamental system of solution. Fundamental system of solutions must exist.

General solution of system (1.36) may be presented in the form of a linear combination with arbitrary constant coefficients of particular solutions composing a fundamental system.

§ 7. Transformation of an Equation of Free Oscillations into a System of Linear First Order Differential Equations

For convenience of analysis of free oscillations presented by equation (1.1),

It frequently turns out to be expedient to replace equation (0.1) by an equivalent system of linear first order differential equations. With this goal, it is possible either to return to system (1.29) from which is obtained equation (0.1) or to cross from variable x, which is the unknown variable of the latter, to a system of new variables $x_1, \ldots, x_n$ with the help of nonsingular[1] (for all t) linear substitution

$$\left.\begin{aligned} x &= c_{11}x_1 + \ldots + c_{1n}x_n, \\ \frac{dx}{dt} &= c_{21}x_1 + \ldots + c_{2n}x_n, \\ &\cdots\cdots\cdots\cdots \\ \frac{d^{n-1}x}{dt^{n-1}} &= c_{n1}x_1 + \ldots + c_{nn}x_n, \end{aligned}\right| \tag{1.37}$$

where $c_{ij} = c_{ij}(t)$ are any continuous, differentiable functions from t, real or complex-valued.

Since the first method, returning to initial system of equations, does not facilitate problem of analysis of oscillations, we will center our attention on the second method.

Differentiating equations of system (1.37), we will obtain

$$\left.\begin{aligned} \frac{dx}{dt} &= c_{11}\dot{x}_1 + \ldots + c_{1n}\dot{x}_n + \dot{c}_{11}x_1 + \ldots + \dot{c}_{1n}x_n, \\ \frac{d^2x}{dt^2} &= c_{21}\dot{x}_1 + \ldots + c_{2n}\dot{x}_n + \dot{c}_{21}x_1 + \ldots + \dot{c}_{2n}x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d^nx}{dt^n} &= c_{n1}\dot{x}_1 + \ldots + c_{nn}\dot{x}_n + \dot{c}_{n1}x_1 + \ldots + \dot{c}_{nn}x_n. \end{aligned}\right| \tag{1.38}$$

Having compared i-th equation of system (1.38) with i + 1-st equations of system (1.37), and n-th equation of system (1.38) with equation (0.1), we will find system of equations, which variables $x_1, \ldots, x_n$ satisfy. It has the form

$$\left.\begin{aligned} c_{11}\dot{x}_1 + \ldots + c_{1n}\dot{x}_n &= (c_{21} - \dot{c}_{11})x_1 + \ldots + (c_{2n} - \dot{c}_{1n})x_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{n-1,1}\dot{x}_1 + c_{n-1,n}\dot{x}_n &= (c_{n1} - \dot{c}_{n-1,1})x_1 + \ldots + (c_{nn} - \dot{c}_{n-1,n})x_n, \\ c_{n1}\dot{x}_1 + \ldots + c_{nn}\dot{x}_n &= -b_1(c_{n1}x_1 + \ldots + c_{nn}x_n) - \ldots \\ &\ldots - b_n(c_{11}x_1 + \ldots + c_{1n}x_n) - \dot{c}_{n1}x_1 - \ldots - \dot{c}_{nn}x_n. \end{aligned}\right| \tag{1.39}$$

Considering this system as a system of linear algebraic equations relative to unknowns $x_1, \ldots, x_n$ and solving it relative to these unknowns (which is always possible since determinant det $\|c_{ij}\|$ is different than zero), we will obtain a system of differential equations of the form

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[1] Linear substitution (1.37) is called <u>nonsingular</u>, if

$$\det \| c_{ij} \|_1^n \neq 0.$$

$$\dot{x}_i = d_{i1}x_1 + \ldots + d_{in}x_n \quad (i = 1, 2, 3, \ldots, n). \tag{1.40}$$

System (1.40) and equation (0.1) are equivalent: because of equations (1.37), between their solutions there exists a one-to-one conformity. Formulas of transition from one solution to an other depend on coefficient of system (1.37) and have simple form.

The above-stated determines method of construction of system of linear first order differential equations, equivalent to investigated equation of free oscillations. Since selection of coefficients c_{ij} of equations (1.37), to a great degree, is arbitrary, by modifying these coefficients there can be obtained different systems (1.40) whose coefficients, corresponding to each other in indices, essentially differ.

There appears the question: what functions are expedient to select as coefficients $c_{ij}(t)$?

Since final result of transformations is system (1.40), then the stake in the search for an answer to the set question must be that form of system (1.40) which is the most desirable. Obviously, the problem of analysis of free oscillations would be basically solved if substitutuion (1.37) brought equation (0.1) to system (1.40) with diagonal or triangular matrix of coefficients. Consequently, during selection of coefficients c_{ij} it is necessary to try to have matrix of coefficients of system (1.40), in some meaning, close to matrix of diagonal or triangular form.

Many different methods are possible to determine coefficients c_{ij} satisfying mentioned requirement. Such methods, in particular, include methods connected with transforming the equation of free oscillations with the help of special (called by us canonical) expansions of the solution of the equation of free oscillations. These methods are presented in the following chapter.

C H A P T E R II

CANONICAL EXPANSIONS OF SOLUTION OF AN EQUATION OF FREE OSCILLATIONS

§ 1. Local Approximation to a Solution of an Equation of Free Oscillations

Let us assume that T, t_0 and $t_0 + \Delta t$ are certain fixed moments of time, connected by relationship

$$T < t_0 < t_0 + \Delta t,$$

and let us assume that roots of algebraic equation

$$\lambda^n + b_1(t)\lambda^{n-1} + \ldots + b_n(t) = 0 \qquad (2.0)$$

(in which t is considered as a parameter) are different during $t = t_0$. We will constitute equation

$$\frac{d^n x^*}{dt^n} + b_1^* \frac{d^{n-1} x^*}{dt^{n-1}} + \ldots + b_n^* x^* = 0, \qquad (2.1)$$

after determining coefficients b_i^* $(i = 1, 2, \ldots, n)$ in section $[T, t_0 + \Delta t]$ in the following manner:

$$b_i^*(t) = b_i(t) \qquad (T \leqslant t \leqslant t_0),$$
$$b_i^*(t) = b_i(t_0) \qquad (t_0 \leqslant t \leqslant t_0 + \Delta t),$$

where b_i $(i = 1, 2, \ldots, n)$ coefficients of equation (0.1).

Considering initial conditions

$$\left.\begin{aligned} x^*(t_0) &= x(t_0) = \xi_0, \\ \left(\frac{dx^*}{dt}\right)_{t=t_0} &= \left(\frac{dx}{dt}\right)_{t=t_0} = \xi_1, \\ &\cdots\cdots\cdots\cdots \\ \left(\frac{d^{n-1}x^*}{dt^{n-1}}\right)_{t=t_0} &= \left(\frac{d^{n-1}x}{dt^{n-1}}\right)_{t=t_0} = \xi_{n-1}, \end{aligned}\right| \qquad (2.2)$$

common for equations (0.1) and (2.1), we will compare their solution in interval $(t_0, t_0 + \Delta t)$.

Let us assume that

$$\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)} -$$

are values of functions $\lambda_i(t)$ $(i = 1, \ldots, n)$ at $t = t_0$. Then solution of equation (2.1) in interval $(t_0, t_0 + \Delta t)$ may be presented in the form

$$x^*(t) = \sum_{i=1}^{n} C_i e^{(t-t_0)\lambda_i^{(0)}}, \tag{2.3}$$

where C_i are constants satisfying conditions

$$\left.\begin{aligned} &\sum_{i=1}^{n} C_i = x(t_0) = \xi_0, \\ &\sum_{i=1}^{n} \lambda_i^{(0)} C_i = \left(\frac{dx}{dt}\right)_{t=t_0} = \xi_1, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\sum_{i=1}^{n} (\lambda_i^{(0)})^{n-1} C_i = \left(\frac{d^{n-1}x}{dt^{n-1}}\right)_{t=t_0} = \xi_{n-1}. \end{aligned}\right| \tag{2.4}$$

We will introduce variables

$$x_1 = x,\ x_2 = \frac{dx}{dt}, \ldots, x_n = \frac{d^{n-1}x}{dt^{n-1}},$$

$$x_1^* = x^*,\ x_2^* = \frac{dx^*}{dt}, \ldots, x_n^* = \frac{d^{n-1}x^*}{dt^{n-1}}$$

and will replace equations (0.1) and (2.1) by equivalent systems of equations

$$\left.\begin{aligned} &\dot{x}_1 = x_2, \\ &\dot{x}_2 = x_3, \\ &\dot{x}_n = -b_n x_1 - b_{n-1} x_2 - \ldots - b_1 x_n \end{aligned}\right| \tag{2.5}$$

and

$$\left.\begin{aligned} &\dot{x}_1^* = x_2^*, \\ &\dot{x}_2^* = x_3^*, \\ &\ldots\ldots\ldots\ldots\ldots \\ &\dot{x}_n^* = -b_n^* x_1^* - b_{n-1}^* x_2^* - \ldots - b_1^* x_n^*. \end{aligned}\right| \tag{2.6}$$

Assuming continuity of coefficients $b_i(t)$, it is always possible to indicate such positive constants ε and B, so that in interval $(t_0, t_0 + \Delta t)$ are executed inequalities

$$\left.\begin{aligned} &|b_i| < B, \\ &|b_i - b_i^*| < \varepsilon \\ &(i = 1, 2, \ldots, n). \end{aligned}\right| \tag{2.7}$$

During conditions (2.7), on the basis of one of the theorems of approximate [26] solution for a system of linear differential equations the following inequality is

valid

$$\sum_{i=1}^{n}|x_i(t) - x_i^*(t)| < \frac{\varepsilon}{B}\sum_{i=1}^{n}|x_i(t_0)|\,e^{(2B+1)\Delta t} = \frac{\varepsilon}{B}\sum_{i=0}^{n-1}|\xi_i|\,e^{(2B+1)\Delta t} \tag{2.8}$$
$$(t_0 < t < t_0+\Delta t).$$

From inequality (2.8) there ensues the following system of inequalities,

$$\left.\begin{aligned}
&|x - x^*| < \frac{\varepsilon}{B}\sum_{i=0}^{n-1}|\xi_i|\,e^{(2B+1)\Delta t},\\
&\left|\frac{dx}{dt} - \frac{dx^*}{dt}\right| < \frac{\varepsilon}{B}\sum_{i=0}^{n-1}|\xi_i|\,e^{(2B+1)\Delta t},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\left|\frac{d^{n-1}x}{dt^{n-1}} - \frac{d^{n-1}x^*}{dt^{n-1}}\right| < \frac{\varepsilon}{B}\sum_{i=0}^{n-1}|\xi_i|\,e^{(2B+1)\Delta t}.\\
&(t_0 < t < t_0+\Delta t)
\end{aligned}\right| \tag{2.9}$$

It follows from this that for a given value $t = t_0$ and given positive constant η it is always possible to indicate such a value of $t = t_0 + \Delta t$ that at any point of the interval $(t_0, t_0 + \Delta t)$ difference of values of functions $x^*(t)$ and $x(t)$, and also differences in values of their derivatives from the first to n-1-st, inclusively, does not exceed η.

Because of equation (2.3), from that proven it follows that for mentioned given numbers t_0 and η it is always possible to indicate such number $t_0 + \Delta t$ $(\Delta t > 0)$, that there will be executed inequalities[1]

$$\left|\frac{d^k x}{dt^k} - \sum_{i=1}^{n}(\lambda_i^{(0)})^k C_i e^{(t-t_0)\lambda_i^{(0)}}\right| < \eta \tag{2.10}$$
$$(k=0,\ 1,\ldots,\ n-1)$$

for any values of t from interval $(t_0, t_0 + \Delta t)$. Consequently, there always exists finite interval $(t_0, t_0 + \Delta t)$, in which solution of equations (0.1) allows an approximate presentation in the form

$$\frac{d^k x}{dt^k} \approx \sum_{i=1}^{n}(\lambda_i^{(0)})^k C_i e^{(t-t_0)\lambda_i^{(0)}} \tag{2.11}$$
$$(k=0,\ 1,\ldots,\ n-1).$$

where maximum interval for which its validity is guaranteed is determined by inital

[1] Here and subsequently under zero derivative of arbitrary magnitude u with respect to t, there is understood this actual magnitude, i.e., there is implied equality

$$\frac{d^0 u}{dt^0} = u.$$

conditions and given accuracy of approximation in accordance with inequalities (2.9).

§ 2. Approximation of Solution of an Equation of Free Oscillations in a Given Finite Interval

Formulas (2.11) for approximation of a solution of an equation of free oscillations are valid for a certain rather small interval (t_0, $t_0 + \Delta t$), adjoining, on the right, point t_0. If, besides accuracy of approximation, is given also interval (t_0, $t_0 + \Delta t$), then formulas (2.11), in general, are incorrect. Approximate presentation of a solution of an equation of free oscillations for a given finite interval can be obtained on the basis of the theorem of S. M. Alferov [26].

Theorem of Alferov. Solution of system (2.5) may be, with some degree of accuracy, obtained in a given finite interval by means of dividing the latter into a finite number of equal subintervals and by replacing variables of coefficients inside each interval by constants equal to any values of corresponding variables of coefficients inside or on boundaries of considered subintervals.[1]

Thus, on the basis of the given theorem, in order to obtain approximate solution of system (2.5) in a given interval (t_0, $t_0 + \Delta t$) during arbitrarily given initial conditions $x_i(t_0) = \xi_{i-1}$ ($i = 1, \ldots, n$), it is sufficient to break down interval (t_0, $t_0 + \Delta t$) into a sufficiently large number m (determined by required degree of approximation) of equal subintervals (t_0, $t^{(1)}$), ($t^{(1)}$, $t^{(2)}$), ..., ($t^{(m-1)}$, $t^{(m)}$), where $t^{(m)} = t_0 + \Delta t$), and to solve system of equations

$$\left.\begin{aligned} \dot{\tilde{x}}_1 &= \tilde{x}_2, \\ \dot{\tilde{x}}_2 &= \tilde{x}_3, \\ &\cdots\cdots\cdots\cdots\cdots \\ \dot{\tilde{x}}_n &= -\tilde{b}_n \tilde{x}_1 - \tilde{b}_{n-1}\tilde{x}_2 - \ldots - \tilde{b}_1 \tilde{x}_n, \end{aligned}\right\} \tag{2.12}$$

where for every $i = 1, \ldots, n$ $\tilde{b}_i(t) = b_i(t)$ $(t \leq t_0)$.

$$\tilde{b}_i(t) = b_i(t^{(k-1)}) \; (t^{(k-1)} \leqslant t < t^{(k)}, \; k = 1, \ldots, n, \; t^{(0)} = t_0)$$ during

initial conditions $\tilde{x}_i(t) = \xi_{i-1}$ ($i = 1, 2, \ldots, n$). Solution of this system it is possible to consider as approximate solution of system (2.5).

By excluding variables $\tilde{x}_2, \ldots, \tilde{x}_n$, system (2.12) is reduced to equation

[1] In formulation of the theorem given by S. M. Alferov, the possibility of equating constant coefficients to values of variable coefficients on the boundaries of considered intervals is not mentioned; however, such possibility follows from the proof of the theorem, given in work [26].

$$\frac{d^n\tilde{x}}{dt^n} + \tilde{b}_1 \frac{d^{n-1}\tilde{x}}{dt^{n-1}} + \dots + \tilde{b}_n \tilde{x} = 0, \tag{2.13}$$

where

$$\tilde{x} = \tilde{x}_1, \quad \frac{d\tilde{x}}{dt} = \tilde{x}_2, \dots, \frac{d^{n-1}\tilde{x}}{dt^{n-1}} = \tilde{x}_n.$$

Owing to the above-considered relationship between particular solutions of systems (2.5) and (2.12) determined by identical initial conditions, the particular solution of equation (2.13) and its first n - 1 derivative during arbitrary initial conditions can be considered as functions approximating solution (corresponding to it in initial conditions) of equation (0.1) and n - 2 derivative of this solution.

Solution of equation (2.13) in interval $(t_0, t_0 + \Delta t)$ during absence of multiple roots of equation (2.0) during initial conditions

$$\tilde{x} = \xi_0, \quad \frac{d\tilde{x}}{dt} = \xi_1, \dots, \frac{d^{n-1}\tilde{x}}{dt^{n-1}} = \xi_{n-1}$$

may be built in the following form.

Let us find solution of equation (2.13) in interval $(t_0, t^{(1)})$; as it was shown in the preceding paragraph, it may be recorded in the form

$$\tilde{x}(t) = \sum_{i=1}^{n} C_i e^{(t-t_0)\lambda_i^{(0)}}, \tag{2.14}$$

where C_i are constants satisfying conditions (2.4).

For derivatives of solution we will find

$$\frac{d^k\tilde{x}}{dt^k} = \sum_{i=1}^{n} (\lambda_i^{(0)})^{k} C_i e^{(t-t_0)\lambda_i^{(0)}}. \tag{2.15}$$
$$(k = 1, \dots, n-1)$$

We will define value of function $\tilde{x}(t)$ and derivatives $\frac{d\tilde{x}}{dt}, \dots, \frac{d^{n-1}\tilde{x}}{dt^{n-1}}$ at the end of considered interval and, using them as initial values of corresponding magnitudes in interval $(t^{(1)}, t^{(2)})$, we will find solution of equation (2.13) in this interval. We will write it in the form

$$\tilde{x}(t) = \sum_{i=1}^{n} C_i^{(1)} e^{(t-t^{(1)})\lambda_i(t^{(1)})}, \tag{2.16}$$

where $C_i^{(1)}$ are constants satisfying conditions

$$\sum_{i=1}^{n} C_i^{(1)} = \tilde{x}(t^{(1)}),$$

$$\sum_{i=1}^{n} \lambda_i(t^{(1)}) C_i^{(1)} = \left(\frac{d\tilde{x}}{dt}\right)_{t=t^{(1)}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} [\lambda_i(t^{(1)})]^{n-1} C_i^{(1)} = \left(\frac{d^{n-1}\tilde{x}}{dt^{n-1}}\right)_{t=t^{(1)}}.$$

Derivatives of solution $\tilde{x}(t)$ in interval $(t^{(1)}, t^{(2)})$ have the form

$$\frac{d^k\tilde{x}}{dt^k} = \sum_{i=1}^{n} [\lambda_i(t^{(1)})]^{k-1} C_i^{(1)} e^{(t-t^{(1)})\lambda_i(t^{(1)})} \quad (k=1, \ldots, n-1). \tag{2.17}$$

Continuing the process, we will reach interval $(t^{m-1}, t^{(m)})$, in which solution of equation, obviously, can be recorded in the form

$$\tilde{x}(t) = \sum_{i=1}^{n} C_i^{(m-1)} e^{(t-t^{(m-1)})\lambda_i(t^{(m-1)})}, \tag{2.18}$$

and formula for derivatives of solutions have the form

$$\frac{d^k\tilde{x}}{dt^k} = \sum_{i=1}^{n} [\lambda_i(t^{(m-1)})]^{k-1} C_i^{(m-1)} e^{(t-t^{(m-1)})\lambda_i(t^{(m-1)})}, \quad (k=1, \ldots, n-1), \tag{2.19}$$

where $C_i^{(m-1)}$ $(i = 1, \ldots, n)$ are constants satisfying conditions

$$\sum_{i=1}^{n} \lambda_i(t^{(m-1)}) C_i^{(m-1)} = \left(\frac{d\tilde{x}}{dt}\right)_{t=t^{(m-1)}}, \quad \sum_{i=1}^{n} C_i^{(m-1)} = \tilde{x}(t^{(m-1)}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{i=1}^{n} [\lambda_i(t^{(m-1)})]^{n-1} C_i^{(m-1)} = \left(\frac{d^{n-1}\tilde{x}}{dt^{n-1}}\right)_{t=t^{(m-1)}}.$$

Having compared formulas (2.14)-(2.19), we will find that solution of equation (2.12) and its derivatives in the whole interval $(t_0, t_0 + \Delta t)$ can be presented by formulas

$$\left.\begin{aligned} \tilde{x}(t) &= \sum_{i=1}^{n} \tilde{C}_i(t) e^{(t-\tau(t))\tilde{\lambda}_i(t)}, \\ \frac{d^k\tilde{x}}{dt^k} &= \sum_{i=1}^{n} [\tilde{\lambda}_i(t)]^{k-1} \tilde{C}_i(t) e^{(t-\tau(t))\tilde{\lambda}_i(t)} \\ &(k=1, \ldots, n-1). \end{aligned}\right| \tag{2.20}$$

where $\tilde{C}_i(t)$, $\tau(t)$ and $\tilde{\lambda}_i(t)$ are step functions satisfying conditions

$$\left.\begin{aligned} &\tilde{C}_i(t) = C_i^{(k-1)}, \\ &\tau(t) = t^{(k-1)}, \\ &\tilde{\lambda}_i(t) = \lambda_i(t^{(k-1)}). \end{aligned}\right| \quad \begin{pmatrix} t^{(k-1)} \leq t < t^{(k)} \\ k = 1, \ldots, m \end{pmatrix}.$$

On the basis of the theorem of Alferov during any given positive value of η, interval $(t_0, t_0 + \Delta t)$ can always be broken down into such a finite sequence of elementary intervals that the following inequality will hold:

$$\begin{gathered} |x - \tilde{x}| < \eta, \\ \left| \frac{d^k x}{dt^k} - \frac{d^k \tilde{x}}{dt^k} \right| < \eta \\ \begin{pmatrix} t_0 \leq t < t_0 + \Delta t, \\ k = 1, \ldots, n-1 \end{pmatrix}; \end{gathered}$$

Consequently, any given finite interval $(t_0, t_0 + \Delta t)$ can always be broken down into a finite number of equal subintervals at which solution of equation (0,1) and its derivatives $\frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}$ allow, in this interval, approximate presentations in the form

$$\left.\begin{aligned} &x(t) \approx \sum_{i=1}^{n} \tilde{C}_i(t) e^{(t-\tau)\tilde{\lambda}_i(t)}, \\ &\frac{d^k x}{dt^k} \approx \sum_{i=1}^{n} [\tilde{\lambda}_i(t)]^{k-1} \tilde{C}_i(t) e^{(t-\tau)\tilde{\lambda}_i(t)} \\ &(k = 1, \ldots, n-1). \end{aligned}\right\} \qquad (2.21)$$

where $\tilde{C}_1(t)$, $\tau(t)$ and $\tilde{\lambda}_1(t)$ are step $(t_0 \leq t < t_0 + \Delta t)$ functions, the process of construction of which is described above, the minimum number of subintervals at which is guaranteed the validity of such presentation is determined by initial conditions and given accuracy of approximation. Corresponding numerical ratios can be obtained from relationships given in above-quoted work [26].

§ 3. First Form of Canonical Expansions of Solution of Equation of Oscillations

As follows from Alferov's theorem, it is always possible to designate m, so large that magnitude η will be as small as desired. Therefore, directing η to zero, in accordance with formulas (2.21) we will obtain

$$\left.\begin{aligned}
x(t) &= \lim_{\eta\to 0}\sum_{i=1}^{n}\tilde{C}_i(t)e^{(t-\eta)\tilde{\lambda}_i(t)},\\
\frac{dx(t)}{dt} &= \lim_{\eta\to 0}\sum_{i=1}^{n}\tilde{\lambda}_i(t)\tilde{C}_i(t)e^{(t-\eta)\tilde{\lambda}_i(t)}\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{d^{n-1}x(t)}{dt^{n-1}} &= \lim_{\eta\to 0}\sum_{i=1}^{n}\tilde{\lambda}_i^{(n-1)}(t)\tilde{C}_i(t)e^{(t-\eta)\tilde{\lambda}_i(t)}\\
&(t_0<t<t_0+\Delta t).
\end{aligned}\right| \tag{2.22}$$

We will introduce into consideration magnitudes $\tilde{y}_1(t)$, ..., $\tilde{y}_n(t)$, after determining them by equations

$$\tilde{y}_i(t)=\tilde{C}_i(t)e^{(t-\eta)\tilde{\lambda}_i(t)} \quad (i=1,\ldots,n). \tag{2.23}$$

We will prove that at $\eta\to 0$ limits of these magnitudes exist.

Really, during $t_0 \leq t < t_0+\Delta t$, magnitudes $\tilde{y}_1$, ..., $\tilde{y}_n$ satisfy system of equations

$$\left.\begin{aligned}
&\tilde{y}_1+\ldots+\tilde{y}_n=\tilde{x},\\
&\tilde{\lambda}_1\tilde{y}_1+\ldots+\tilde{\lambda}_n\tilde{y}_n=\frac{d\tilde{x}}{dt}.\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&\tilde{\lambda}_1^{n-1}\tilde{y}_1+\ldots+\tilde{\lambda}_n^{n-1}\tilde{y}_n=\frac{d^{n-1}\tilde{x}}{dt^{n-1}}.
\end{aligned}\right| \tag{2.24}$$

Determinant of system

$$\Delta=\begin{vmatrix} 1 & \ldots & 1\\ \tilde{\lambda}_1 & \ldots & \tilde{\lambda}_n\\ \ldots & \ldots & \ldots\\ \tilde{\lambda}_1^{n-1} & \ldots & \tilde{\lambda}_n^{n-1}\end{vmatrix} \text{-- is known determinant of Vandermonde;} \tag{2.25}$$

it is equal to product

$$(\tilde{\lambda}_n-\tilde{\lambda}_1)(\tilde{\lambda}_n-\tilde{\lambda}_2)(\tilde{\lambda}_n-\tilde{\lambda}_3)\ldots(\tilde{\lambda}_n-\tilde{\lambda}_{n-1})\ldots(\tilde{\lambda}_3-\tilde{\lambda}_2)(\tilde{\lambda}_2-\tilde{\lambda}_1)$$

and, consequently, because of assumed difference of roots λ_1, ..., λ_n is different than zero.

From condition

$$\Delta\neq 0$$

follows the solvability of system (2.24) relative to magnitudes $\tilde{y}_1$, ..., $\tilde{y}_n$.

After solving this system, we will find formulas expressing shown magnitudes $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n; \tilde{x}, \frac{d\tilde{x}}{dt}, \ldots, \frac{d^{n-1}\tilde{x}}{dt^{n-1}}$. Considering that limits of the latter exist, on the basis of known theorems from the theory of limits we will conclude that limits of magnitude $\tilde{y}_1, \ldots, \tilde{y}_n$ also exist.

We will designate limits of magnitudes $\tilde{y}_1, \ldots, \tilde{y}_n$ by symbols $y_1, \ldots, y_n$, i.e.,

$$\lim_{q\to\infty} \tilde{y}_i = y_i \quad (i = 1, \ldots, n). \tag{2.26}$$

Considering that

$$\lim_{q\to\infty} \tilde{\lambda}_i = \lambda_i \quad (i = 1, \ldots, n), \quad \lim_{q\to\infty} \tilde{x} = x,$$

$$\lim_{q\to\infty} \frac{d\tilde{x}}{dt} = \frac{dx}{dt}, \ldots, \lim_{q\to\infty} \frac{d^{n-1}\tilde{x}}{dt^{n-1}} = \frac{d^{n-1}x}{dt^{n-1}},$$

we will find that they satisfy system of equations

$$\begin{aligned} &y_1 + \ldots + y_n = x, \\ &\lambda_1 y_1 + \ldots + \lambda_n y_n = \frac{dx}{dt}, \\ &\cdots\cdots\cdots\cdots \\ &\lambda_1^{n-1} y_1 + \ldots + \lambda_n^{n-1} y_n = \frac{d^{n-1}x}{dt^{n-1}}. \end{aligned} \tag{2.27}$$

System (2.27), obviously, is soluble relative to magnitudes $y_1, \ldots, y_n$.

In the future we will be interested in magnitudes y_1, but not $\tilde{y}_1$. Equalities (2.26) establish their analytic meaning; nowhere further are they used. For determination of the connection of these magnitudes with solution (0.1), subsequently there is applied only system (2.27).

Equations (2.27) place the following set of functions in unique correspondence to each solution of equation (0.1) $x(t)$ in interval $(t_0, t_0 + \Delta t)$.

$$y_1(t), \ldots, y_n(t).$$

Obviously, values of these functions at any point of the interval do not depend on length of interval.

On the basis of the first equation of system (2.27) sum $y_1(t) + \ldots + y_n(t)$ coincides with solution $x(t)$:

$$x = y_1 + \ldots + y_n. \tag{2.28}$$

Presentation of solution of equation of free oscillations in the form of sum of functions $y_1, \ldots, y_n$, satisfying condition (2.27), we will call <u>canonical</u>

expansion of solution of equation of free oscillations. Application of this term is justified by the fact that during constant coefficients $b_1, \ldots, b_n$ such expansion coincides with known expansion of solution of equation (0.1), reducing this equation to a system of "canonical form" [15].

Let us agree also to call functions $y_1(t), \ldots, y_n(t)$ canonical components of solution $x(t)$.

Canonical components of solution can be both real and complex functions t, even if solution $x(t)$ is real. The case of real solution $x(t)$ is especially interesting since in practical applications of the theory of free oscillations, real solutions namely usually present interest. We will pause on this in greater detail.

Let us assume that $\lambda_1, \lambda_2, \ldots, \lambda_{n-2m}$ (where m is an integer, smaller than or equal to number n/2) are real roots of equation (2.0); λ_{n-2m+1} and $\lambda_{n-2m+2}, \ldots, \lambda_{n-1}$ and λ_n are pairs of conjugate, complex roots. Then on the basis of equations (2.27), it is simple to establish that for real solution $x(t)$ canonical components $y_1(t), \ldots, y_{n-2m}(t)$ are real, and canonical components $y_{n-2m+1}(t), \ldots, y_n(t)$ are complex, where magnitudes $y_{n-2m+1}, y_{n-2m+2}, \ldots, y_{n-1}$ and y_n are conjugate, i.e., that, during real $x(t)$, real roots correspond to real canonical components and conjugate complex roots correspond to conjugate complex canonical components.

Really, solving system (2.27) relative to canonical components y_1, we will obtain

$$y_1 = \frac{\begin{vmatrix} x & 1 & \ldots & 1 \\ \dot{x} & \lambda_2 & \ldots & \lambda_n \\ \cdot & \cdot & \cdot & \cdot \\ x^{(n-1)} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \cdot & \cdot & \cdot \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}} \qquad (2.29)$$

Let us assume that λ_1 is a real root. Then, in order that component y_1 be real, it is necessary and sufficient to execute condition

$$y_1 = \bar{y}_1.$$

We will satisfy ourselves that this condition indeed is executed. With this goal, let us note that transition in the right part of formula (2.29) to complex conjugate magnitudes may be carried out with the help of transposition (in determinant

of numerator and denominator) of columns corresponding to conjugate complex roots. This does not lead to change of numerator and denominator if m is even, and changes their signs if m is odd. Consequently, magnitude of the right side, on the whole, remains constant, and hence $y_1 = \bar{y}_1$.

Obviously, the given reasonings, in equal measure, are valid for all other real roots.

Let us assume now that roots λ_{n-1} and λ_n are conjugate complex, and we will solve system (2.27) relative to canonical components y_{n-1} and y_n. We will obtain

$$\left.\begin{aligned} y_{n-1} &= \frac{\begin{vmatrix} 1 & \ldots & x & 1 \\ \lambda_1 & \ldots & \frac{dx}{dt} & \lambda_n \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_1^{n-1} & \ldots & \frac{d^{n-1}x}{dt^{n-1}} & \lambda_n^{n-1} \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \cdot & \cdot & \cdot \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}} \\ y_n &= \frac{\begin{vmatrix} 1 & \ldots & 1 & x \\ \lambda_1 & \ldots & \lambda_{n-1} & \frac{dx}{dt} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_1^{n-1} & \ldots & \lambda_{n-1}^{n-1} & \frac{d^{n-1}x}{dt^{n-1}} \end{vmatrix}}{\begin{vmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{vmatrix}} \end{aligned}\right\} \qquad (2.30)$$

We will cross, in the right part of the first formula, to complexly conjugate magnitudes. The new denominator will be equal to the old multiplied by $(-1)^m$, and the numerator will be turned into the numerator of the right side of the second formula also with cofactor $(-1)^m$. Consequently, a magnitude complexly conjuate with all the right side of the first formula will coincide with the right side of the second formula, and hence

$$\bar{y}_{n-1} = y_n.$$

Obviously, the same property is possessed by other pairs of canonical components corresponding to conjugate complex roots.

We will consider that functions $b_1(t), \ldots, b_n(t)$ not only are continuous but

also are differentiable. With this, functions $\lambda_1(t)$, ..., $\lambda_n(t)$ also are differentiable. Differentiating, term by term, equations of system (2.27), we will obtain

$$\left.\begin{array}{c} \dot{y}_1+\ldots+\dot{y}_n=\frac{dx}{dt}, \\ \lambda_1\dot{y}_1+\ldots+\lambda_n\dot{y}_n+\dot{\lambda}_1 y_1+\ldots+\dot{\lambda}_n y_n=\frac{d^2x}{dt^2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1^{n-1}\dot{y}_1+\ldots+\lambda_n^{n-1}\dot{y}_n+ \\ +(n-1)\lambda_1^{n-2}\dot{\lambda}_1 y_1+\ldots+(n-1)\lambda_n^{n-2}\dot{\lambda}_n y_n=\frac{d^nx}{dt^n}. \end{array}\right\} \tag{2.31}$$

After supplementing system (2.27) by equation

$$\lambda_1^n y_1+\lambda_2^n y_2+\ldots+\lambda_n^n y_n=\frac{d^nx}{dt^n}, \tag{2.32}$$

which follows from system (2.27), equation (0.1) and equation (2.0), and having compared obtained system with system (2.31), we will find a system of differential equations, which canonical components y_1, ..., y_n satisfy. This system has the form

$$\left.\begin{array}{c} \dot{y}_1+\ldots+\dot{y}_n=\lambda_1 y_1+\ldots+\lambda_n y_n, \\ \lambda_1\dot{y}_1+\ldots+\lambda_n\dot{y}_n=(\lambda_1^2-\dot{\lambda}_1)y_1+\ldots+(\lambda_n^2-\dot{\lambda}_n)y_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1^{n-1}\dot{y}_1+\ldots+\lambda_n^{n-1}\dot{y}_n= \\ =[\lambda_1^n-(n-1)\lambda_1^{n-2}\dot{\lambda}_1]y_1+\ldots+[\lambda_n^n-(n-1)\lambda_n^{n-2}\dot{\lambda}_n]y_n. \end{array}\right\} \tag{2.33}$$

Considering system (2.33) as a system of linear algebraic equations relative to unknowns $\dot{y}_1$, ..., $\dot{y}_n$ and solving it relative to these unknowns, will give it the form

$$\dot{y}_i=\lambda_i y_i+\sum_{j=1}^{n} g_{ij}y_j \quad (i=1,\ldots,n), \tag{2.34}$$

where

$$g_{ij}=-\frac{\dot{\lambda}_j}{W}\left|\begin{array}{ccccccc} 1 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_{i-1} & 1 & \lambda_{i+1} & \ldots & \lambda_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_1^{n-1} & \ldots & \lambda_{i-1}^{n-1} & (n-1)\lambda_j^{n-2} & \lambda_{i+1}^{n-1} & \ldots & \lambda_n^{n-1} \end{array}\right|$$

W is determinant of matrix of coefficients of left part of system (2.27).

In system (2.34) variables y_1, ..., y_n and coefficients λ_j and g_{ij} (i, j = 1, ..., n) are complex-valued functions of t. Considering the above indexation of real and complexly conjugate roots, We will obtain

$$g_{ij}=\ddot{g}_{ij} \quad \text{при } i,\ j \leqslant n-2m,$$

$$\left.\begin{aligned} g_{ij}&=\ddot{g}_{i,j+1} \\ g_{ji}&=\bar{g}_{j+1,i} \end{aligned}\right\} \begin{aligned} &\text{при } i \leqslant n-2m; \\ &j=n-2m+1,\ n-2m+3,\ \dots,\ n-1. \end{aligned}$$

$$\left.\begin{aligned} g_{ij}&=\bar{g}_{i+1,j+1}; \\ g_{i,j+1}&=\ddot{g}_{i+1,j} \end{aligned}\right\} \quad \text{при } i,\ j=n-2m+1,\ n-2m+3,\ \dots,\ n-1.$$

Introducing canonical components $y_1, \dots, y_n$ as a limiting form of functions $\tilde{y}_1, \dots, \tilde{y}_n$, contained in formula of approximation of solution of equation of free oscillations in a given finite interval, we thereby constructed canonical expansion of solution in a finite interval. Digressing from formula of approximation, assuming that conditions of absence of multiple roots of equation (2.0) are executed in half-open interval (T, ∞), and determining formally canonical components as solution of system (2.27) for given functions x, $dx/dt, \dots, d^{n-1}x/dt^{n-1}$ in this interval, we will expand introduced idea of canonical expansion into open interval $t_0, \infty)$.

Necessary and sufficient condition of absence of multiple roots of equation (2.0), during a fixed value of t, has the form

$$W \neq 0. \tag{2.35}$$

Inequality (2.35) is equivalent to inequality

$$W^2 \neq 0.$$

Magnitude W^2, presented as a function of roots of algebraic equation (2.0), is called discriminant of this equation and is calculated by the formula

$$W^2=\begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} \\ s_1 & s_2 & s_3 & \dots & s_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2} \end{vmatrix}, \tag{2.36}$$

where $s_0 = n$,

$$s_\alpha=\sum_{i=1}^{n}\lambda_i^\alpha \quad (\alpha=1,\ 2,\ \dots,\ 2n-2).$$

Function $W^2(\lambda_1, \dots, \lambda_n)$ is polynomial from roots $\lambda_1, \dots, \lambda_n$. This polynomial is symmetric [28], i.e., does not change during any transposition of magnitudes $\lambda_1, \dots, \lambda_n$. Because of the fundamental theorem on symmetric polynomials it may be presented in the form of a polynomial from coefficients $b_1, \dots, b_n$. The latter we can determine using equality (2.36), if for calculation of magnitudes s_α we use, for instance, formula of Varing [27]

$$s_s = s\sum \frac{(-1)^{\mu_1+\mu_2+\mu_3+\cdots}(\mu_1+\mu_2+\ldots+\mu_n-1)!}{\mu_1!\,\mu_2!\ldots\mu_n!} f_1^{\mu_1} f_2^{\mu_2}\ldots f_n^{\mu_n},$$

where sum is taken with respect to all combinations of positive integers or equal to zero of numbers $\mu_1, \mu_2, \ldots, \mu_n$, satisfying condition

$$\mu_1 + 2\mu_2 + \ldots + s\mu_s = s.$$

and magnitudes f_i ($i = 1, \ldots, n$) are connected with coefficients b_i by relationships

$$f_i = (-1)^i b_i.$$

In particular, at $n = 2$, the necessary and sufficient condition of absence of multiple roots of equation (2.0) has the form

$$W^2 = b_1^2 - 4b_2 \neq 0.$$

At $n = 3$, the necessary and sufficient condition of absence of multiple roots has the form

$$W^2 = b_1^2 b_2^2 - 2b_1^3 b_3 + 18 b_1 b_2 b_3 - 4b_2^3 - 27 b_3^2 \neq 0.$$

At $n = 4$, the necessary and sufficient condition of absence of multiple roots has the form

$$\begin{aligned} W^2 = {} & 10 b_1^{12} - 120 b_1^{10} b_2 + 118 b_1^9 b_3 + b_1^8 (541 b_2^2 - 82 b_4) - 1012 b_1^7 b_2 b_3 + \\ & + b_1^6 (-1116 b_2^3 + 455 b_3^2 + 622 b_2 b_4) + b_1^5 (2664 b_2^2 b_3 - 616 b_3 b_4) + \\ & + b_1^4 (1008 b_2^4 - 2114 b_2 b_3^2 - 1436 b_2^2 b_4 + 149 b_4^2) + b_1^3 (-2552 b_2^3 b_3 + \\ & + 548 b_3^3 + 2466 b_2 b_3 b_4) + b_1^2 (2177 b_2^2 b_3^2 - 296 b_2^5 + 932 b_2^3 b_4 - \\ & - 400 b_2 b_4^2 - 1038 b_3^2 b_4) + b_1 (592 b_2^4 b_3 - 606 b_2 b_3^3 - 1696 b_2^2 b_3 b_4 + \\ & + 288 b_3 b_4^2) - 27 b_3^4 - 297 b_2^3 b_3^2 + 768 b_2 b_3^2 b_4 - 64 b_2^2 b_4^2 + 256 b_4^3 \neq 0. \end{aligned}$$

Considered canonical expansion of solution of equation of free oscillations with continuous differentiable coefficients, besides the shown properties, possesses a series of interesting peculiarities. We will pause on some of them.

1. Canonical components are linearly expressed through solution $x(t)$ and its derivatives $\frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}$; this property directly follows from form of system (2.27).

2. Canonical component $y_j(t)$ of solution $x(t)$, which is sum of solutions $x'(t)$ and $x''(t)$, is equal to sum of canonical components $y_j'(t)$ and $y_j''(t)$ of these solutions.

3. Canonical components are continuous differentiable functions of time t.

The last two properties follow from the first.

As an example we will define canonical components of known solution of equation of free oscillations.

<u>Example.</u> In equations of Euler [25]

$$\frac{d^2x}{dt^2} + \frac{c}{t^2} x = 0$$

roots of equation $\lambda^2 + b_2 = 0$ are different during all $t > 0$, if $c > 0$. Equation has solution

$$x = \begin{cases} C_1 t^{\frac{1}{2}+s} + C_2 t^{\frac{1}{2}-s} & \text{when } 1 - 4c > 0 \\ C_1 \sqrt{t} + C_2 \sqrt{t} \ln t & \text{при } 1 - 4c = 0 \\ C_1 \sqrt{t} \cos(s \ln t) + C_2 \sqrt{t} \sin(s \ln t) & \text{when } 1 - 4c < 0, \end{cases}$$

where $s = \frac{1}{2} \sqrt{|1 - 4c|}$.

Formulas of connection between canonical components and solution for second order equation have the form

$$\left.\begin{aligned} y_1 &= \frac{x\lambda_2 - \dot{x}}{\lambda_2 - \lambda_1}, \\ y_2 &= \frac{x\lambda_1 - \dot{x}}{\lambda_1 - \lambda_2}. \end{aligned}\right\}$$

Since in this case

$$\lambda_{1,2} = \pm \frac{i\sqrt{c}}{t},$$

where $i = \sqrt{-1}$, then for y_1 and y_2 we have

$$\left.\begin{aligned} y_1 &= \frac{1}{2}\left(x - \frac{it}{\sqrt{c}} \dot{x}\right), \\ y_2 &= \frac{1}{2}\left(x + \frac{it}{\sqrt{c}} \dot{x}\right). \end{aligned}\right\}$$

In accordance with formulas of solution we will obtain

$$\left.\begin{aligned} y_1 &= \frac{C_1}{2}\left[1 - \frac{i(1+2s)}{2\sqrt{c}}\right] t^{\frac{1}{2}+s} + \frac{C_2}{2}\left[1 - \frac{i(1-2s)}{2\sqrt{c}}\right] t^{\frac{1}{2}-s}, \\ y_2 &= \frac{C_1}{2}\left[1 + \frac{i(1+2s)}{2\sqrt{c}}\right] t^{\frac{1}{2}+s} + \frac{C_2}{2}\left[1 + \frac{i(1-2s)}{2\sqrt{c}}\right] t^{\frac{1}{2}-s}, \end{aligned}\right\} \text{when } 1 - 4c > 0,$$

$$\left.\begin{aligned} y_1 &= \frac{1}{2}(C_1 + C_2 \ln t)(1 - i)\sqrt{t} - iC_2\sqrt{t}, \\ y_2 &= \frac{1}{2}(C_1 + C_2 \ln t)(1 + i)\sqrt{t} + iC_2\sqrt{t}, \end{aligned}\right\} \text{when } 1 - 4c = 0,$$

$$\left.\begin{aligned} y_1 &= \frac{\sqrt{t}}{2}\left[C_1 - \frac{i}{2\sqrt{c}}(C_1 + 2C_2 s)\right]\cos(s \ln t) + \\ &+ \frac{\sqrt{t}}{2}\left[C_2 - \frac{i}{2\sqrt{c}}(C_2 - 2C_1 s)\right]\sin(s \ln t), \\ y_2 &= \frac{\sqrt{t}}{2}\left[C_1 + \frac{i}{2\sqrt{c}}(C_1 + 2C_2 s)\right]\cos(s \ln t) + \\ &+ \frac{\sqrt{t}}{2}\left[C_2 + \frac{i}{2\sqrt{c}}(C_2 - 2C_1 s)\right]\sin(s \ln t). \end{aligned}\right\} \text{when } 1 - 4c < 0$$

§ 4. Second Form of Canonical Expansions of Solution of Equation of Oscillations

Considered in the preceding section, canonical expansion of solution of an equation with differentiable coefficients places system of equations (2.34) in

conformity to equation of free oscillations. Coefficients of this system form a matrix, on the main diagonal of which stand magnitudes $\lambda_i + g_{ii}$, and in the remaining places are placed coefficients g_{ij} $(i \neq j)$. It is not difficult to see that coefficients g_{ij} $(i \neq j)$, during changeability of at least one of the coefficients $b_1, \ldots, b_n$, cannot all, simultaneously, turn into zero.[1] Consequently, in the matrix of system (2.34), outside the main diagonal, there must be elements different from zero or, in brief, this matrix has nondiagonal form. It follows from this that system (2.34) does not allow a fundamental system of particular solutions of the form

$$\begin{array}{cccc} y_1(t), & 0, & \ldots, & 0, \\ 0, & y_2(t), & \ldots, & 0, \\ \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot & \cdot\cdot\cdot \\ 0, & 0, & \ldots, & y_n(t). \end{array}$$

Consequently, canonical components, during whatever conditions are found, cannot form a fundamental system of particular solutions of an equation of free oscillations.

If coefficients of the equation of free oscillations possess stronger properties than differentiability, for instance have differentiable first derivatives, then it is possible to improve construction of canonical expansion in such a way that for a certain class of equations of free oscillations the matrix of coefficients of the system of equations relative to canonical components obtains a diagonal form. In this case, the mentioned system of equations is broken down into n first order equations, each of which contains one variable and gives a particular solution of the equation of free oscillations.

We will construct improved constructions of canonical expansions assuming that coefficients of equations of free oscillations have differentiable derivatives from first to p-th order inclusively or, as a limiting case, differentiable derivatives of all orders.

In order to explain meaning of constructions given below, we will assume that for a certain equation general solution is presented in the form

$$x = C_1 \exp \int \zeta_1 \, dt + \ldots + C_n \exp \int \zeta_n \, dt, \tag{2.37}$$

where $C_1, \ldots, C_n$ are arbitrary constants; $\zeta_1(t), \ldots, \zeta_n(t)$ are $n - 1$ multiple, differentiable, complex-valued functions, whose method of construction, with respect

[1]Because of formulas for coefficients g_{ij}, this could take place only if $\lambda_i = 0$ for all i, which is possible only during constant values of all coefficients $b_1, \ldots, b_n$.

to coefficients of equation (0.1), is shown.

Differentiating equation (2.37) 1, 2, ..., k, ..., n - 1 times we will obtain

$$\frac{dx}{dt} = C_1\zeta_1 \exp\int \zeta_1\, dt + \ldots + C_n\zeta_n \exp\int \zeta_n\, dt,$$

$$\frac{d^2x}{dt^2} = C_1(\dot{\zeta}_1 + \zeta_1^2)\exp\int \zeta_1\, dt + \ldots + C_n(\dot{\zeta}_n + \zeta_n^2)\exp\int \zeta_n\, dt,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^kx}{dt^k} = C_1[(\zeta_1 + D)^{k-1}\zeta_1]\exp\int \zeta_1\, dt + \ldots + C_n[(\zeta_n + D)^{k-1}\zeta_n]\exp\int \zeta_n\, dt,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{d^{n-1}x}{dt^{n-1}} = C_1[(\zeta_1 + D)^{n-2}\zeta_1]\exp\int \zeta_1\, dt + \ldots +$$

$$+ C_n[(\zeta_n + D)^{n-2}\zeta_n]\exp\int \zeta_n\, dt,$$

where $(\zeta_j + D)^{i-1}\zeta_j$ (i, j = 1, ..., n) are expressions of the form

$$\underbrace{(\zeta_j + D)\ldots(\zeta_j + D)}_{i-1 \text{ раз.}}\zeta_j$$

$D = \frac{d}{dt}$ is symbol of differentiation.

After designating

$$C_j \exp\int \zeta_j\, dt = z_j \qquad (j = 1, \ldots, n), \tag{2.38}$$

we will unite found equations and equation (2.37) in system

$$\left.\begin{aligned} x &= z_1 + \ldots + z_n, \\ \frac{dx}{dt} &= \zeta_1 z_1 + \ldots + \zeta_n z_n, \\ \frac{d^2x}{dt^2} &= [(\zeta_1 + D)\zeta_1]z_1 + \ldots + [(\zeta_n + D)\zeta_n]z_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{d^{n-1}x}{dt^{n-1}} &= [(\zeta_1 + D)^{n-2}\zeta_1]z_1 + \ldots + [(\zeta_n + D)^{n-2}\zeta_n]z_n. \end{aligned}\right| \tag{2.39}$$

System (2.39) connects arbitrary solutions of considered equation and its derivatives [to n - 1st inclusively] with its particular solutions $z_1(t)$, ..., $z_n(t)$ of the form (2.38). Passing from given particular equation to general, and taking system (2.39) during the same method of construction of functions $\zeta_1(t)$, ..., $\zeta_n(t)$, after determination of canonical expansion, we will obtain, thus, such a construction of canonical expansion by which, at least for one form of equation with variable coefficients of function $z_1(t)$, ..., $z_n(t)$, there are particular solutions of this equation and will form together a fundamental system of particular solutions. If one were to now digress from method of construction of functions $\zeta_1(t)$, ..., $\zeta_n(t)$, then about

canonical expansion of the form (2.39) it is possible to say that it allows the possibility of such cases at which set of functions $z_1(t), \ldots, z_n(t)$ will form fundamental system of particular solutions of equation (0.1). In this property of canonical expansions of form (2.39) also is included the meaning of below mentioned constructions.

We will define the first canonical expansion of the solution of an equation of oscillations by system of equations (2.39), after putting $\zeta_j(t) = \lambda_j(t)$ $(j = 1, \ldots, n)$:

$$\left.\begin{aligned} x &= z_1 + \ldots z_n, \\ \frac{dx}{dt} &= \lambda_1 z_1 + \ldots + \lambda_n z_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d^{n-1}x}{dt^{n-1}} &= [(\lambda_1 + D)^{n-2}\lambda_1] z_1 + \ldots + [(\lambda_n + D)^{n-2}\lambda_n] z_n. \end{aligned}\right| \qquad (2.40)$$

We will apply this expansion only when determinant of matrix of coefficients of the right side of system (2.40) W_1 is different than zero during all values of t of the considered interval and coefficients during variables z_j are differentiable in this interval. Let us note that for differentiability of coefficients during variables z_j, n - 1-multiple differentiability of roots $\lambda_j(t)$ is necessary and sufficient. Applying formula of Viete [27], we will establish conditions which must be satisfied by coefficients of equation (0.1) so that function $\lambda_j(t)$ possess these properties.

Differentiating left and right side of formulas of Viete, we will obtain

$$\left.\begin{aligned} \dot\lambda_1 + \ldots + \dot\lambda_n &= -\dot b_1 \\ \dot\lambda_1(\lambda_2 + \ldots + \lambda_n) + \ldots + \dot\lambda_n(\lambda_1 + \ldots + \lambda_{n-1}) &= \dot b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots& \\ \dot\lambda_1(\lambda_2 \ldots \lambda_n) + \ldots + \dot\lambda_n(\lambda_1 \ldots \lambda_{n-1}) &= (-1)^n \dot b_n. \end{aligned}\right| \qquad (2.41)$$

This system it is possible to consider as a system of linear algebraic equations relative to unknowns $\dot\lambda_1, \ldots, \dot\lambda_n$. It is soluble relative to these unknowns if determinant

$$\begin{vmatrix} 1 & \ldots & 1 \\ \lambda_2 + \ldots + \lambda_n & \ldots & \lambda_1 + \ldots + \lambda_{n-1} \\ \ldots & \ldots & \ldots \\ \lambda_2 \ldots \lambda_n & \ldots & \lambda_1 \ldots \lambda_{n-1} \end{vmatrix}$$

is different than zero. We will show that this determinant is numerically equal to determinant of Vandermonde

$$W=\begin{vmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \dots & \dots & \dots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix}$$

Actually, with the help of formulas of Viete, it may be reduced to the form

$$(-1)^{n-1}\begin{vmatrix} 1 & \dots & 1 \\ b_1+\lambda_1 & \dots & b_1+\lambda_n \\ \dots & \dots & \dots \\ b_{n-1}+\lambda_1 b_{n-2}+\dots+\lambda_1^{n-1} & \dots & b_{n-1}+\lambda_n b_{n-2}+\dots+\lambda_n^{n-1} \end{vmatrix}$$

After designating last determinant by symbol Y, we will present its second line in the form of the sum of

$$(b_1,\dots,b_1)+(\lambda_1,\dots,\lambda_n).$$

We will obtain two determinant components. The first of them is equal to zero, as a determinant with proportional lines (first and second). Consequently, determinant Y coincides with second determinant component. Presenting this latter in the form of the sum of three determinants with different third lines

$$(b_2,\ \dots,\ b_2),$$
$$(b_1\lambda_1,\dots,\ b_1\lambda_n),$$
$$(\lambda_1^2,\dots,\lambda_n^2),$$

let us note that the first two turn into zeroes, as determinants with proportional lines, etc. Continuing the process, we will find that not turning in zeroes the remainder coincides with determinant Vandermonde W.

Thus, system (2.41) is soluble relative to unknowns $\dot{\lambda}_1, \dots, \dot{\lambda}_n$, if determinant of Vandermonde W is different than zero. But conditions of solvability of system relative to shown magnitudes are equivalent to conditions of differentiability of functions $\lambda_1(t), \dots, \lambda_n(t)$. Hence, it follows that for differentiability of functions $\lambda_j(t)$ $(j = 1, \dots, n)$ it is necessary and sufficient that the following condition be fulfilled:

$$W\neq 0. \tag{2.42}$$

Let us assume that condition (2.42) is executed and coefficients $b_1, \dots, b_n$ are twice differentiable. Then, solving system (2.41) relative to unknowns $\dot{\lambda}_1, \dots, \dot{\lambda}_n$, we will obtain formulas expressing these magnitudes through roots $\lambda_1, \dots, \lambda_n$ and

derivatives of coefficients $b_1, \ldots, b_n$. Differentiating these dependences with respect to t and excluding in right sides magnitudes $\dot{\lambda}_1, \ldots, \dot{\lambda}_n$ according to found formulas, we will obtain formulas expressing second derivatives of roots $\lambda_1, \ldots, \lambda_n$ through values of the roots themselves and first and second derivatives of coefficients $b_1, \ldots, b_n$. These dependences have the character of fractional-rational functions from magnitudes $\lambda_1, \ldots, \lambda_n, \dot{b}_1, \ldots, \dot{b}_n, \ddot{b}_1, \ldots, \ddot{b}_n$; the denominator depends only on roots and is equal to W^2. Consequently, for double differentiability of functions $\lambda_1, \ldots, \lambda_n$ it is sufficient that coefficients $b_1, \ldots, b_n$ be twice differentiable and that condition (2.42) be fulfilled.

Continuing this process, there can be obtained analogous, sufficient conditions of existence of continuous highest derivatives of roots. Executing corresponding constructions and uniting particular results, we will formulate the following general conclusion.

<u>For p-multiple differentiability of roots</u> $\lambda_j(t)$ <u>p-multiple differentiability of coefficients</u> $b_1, \ldots, b_n$ <u>is sufficient and condition (2.42).</u>

In particular, at $p = n - 1$ we will obtain the condition interesting us. Considering the obtained result and returning to the above conditions of application of expansion (2.40), we will find that the latter are equivalent to the following conditions:

a) coefficients $b_1(t), \ldots, b_n(t)$ belongs to class $(n - 1)$-multiple of differentiable functions,

b) $W \neq 0$;

c) $W_1 \neq 0$.

Fulfillment of first condition is checked simply. Feasibility of last two conditions, if first is fulfilled, it is possible to check, without determining roots $\lambda_1, \ldots, \lambda_n$.

Really, the system of inequalities shown in conditions of (b) and (c) is equivalent to the following system:

$$\left.\begin{array}{l} W^2 \neq 0, \\ WW_1 \neq 0. \end{array}\right\}$$

Magnitude W^2, as was indicated in § 3, may be represented in the form of a polynomial from coefficients $b_1, \ldots, b_n$. Therefore, inequality $W^2 \neq 0$ determines parameter spaces $b_1, \ldots, b_n$ a family of open domains, on the boundaries of which equality $W^2 = 0$ is satisfied. If the trajectory of a point with coordinates $b_1, \ldots, b_n$, constructed during change t in the considered interval, lies completely

or partially outside these regions, the considered canonical expansion is inapplicable. In particular, it is inapplicable if the mentioned trajectory passes from one region to another.

Product WW_1 (after replacement in expression for determinant W_1 of first and highest derivatives of functions $\lambda_1(t), \ldots, \lambda_n(t)$ by corresponding functions of these magnitudes) of coefficients $b_1, \ldots, b_n$ and their first and highest derivatives (method of obtaining necessary dependence is presented above) may be presented in the form of fractional-rational function of roots $\lambda_1, \ldots, \lambda_n$, the coefficients of which are polynomials from derivatives of coefficients $b_1, \ldots, b_n$ starting with the first and up to (n - 2)nd inclusively.[1] This function possesses such a property that it is not changed during any transposition of magnitudes $\lambda_1, \ldots, \lambda_n$, i.e., it is symmetric [28]. In accordance with the theory of symmetric functions [27], it may be presented in the form of a fraction, the numerator and denominator of which are polynomials from magnitudes $b_1, \ldots, b_n$, where coefficients of these polynomials do not depend on roots $\lambda_1, \ldots, \lambda_n$ and are integers.[2] The denominator of such a fraction, if it is different than unity, can contain only positive integers of the degree[3] of magnitude W^2.

Revealed dependence of product WW_1 on coefficients of equation (0.1) allows us to make the following conclusion. Inequality $WW_1 \neq 0$ determines in n(n - 1)-dimensional space of magnitudes $b_1, \ldots, b_n, \dot{b}_1, \ldots, \dot{b}_n, \ldots, \frac{d^{n-2}b_1}{dt^{n-2}}, \ldots, \frac{d^{n-2}b_n}{dt^{n-2}}$ a family of open domains, so that in all cases when the trajectory of a point with shown coordinates, constructed during change of t in the considered interval, lies completely or partially outside these regions, the considered canonical expansion is inapplicable. Condition $WW_1 = 0$, obviously, is an equation of boundaries of these regions.

Formulas of dependence of magnitude W^2 on coefficients of equation (0.1) in particular cases, during n = 2 and n = 3, were given in the preceding paragraph. Formulas of dependence of magnitude WW_1 on coefficients and derivatives of

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[1] For a second order equation this function is polynomial $(\lambda_2 - \lambda_1)^2$. At n = 3, after certain simplifications, this function also is turned into a polynomial.

[2] Corresponding transformations can be carried out with the help of the formula of Varing given in the preceding paragraph.

[3] Thus, for instance, for n = 4 this degree does not exceed unity, for n = 5 does not exceed three, and for n = 6 does not exceed five.

coefficients of equation (0.1) in the same cases have the form

a) for n = 2

$$W_1 W = W^2 = b_1^2 - 4b_2.$$

b) for n = 3

$$\begin{aligned} W'_1 W = W^2 + 2\dot{b}_1(b_2^2 - 3b_1 b_3) - \dot{b}_2(b_1 b_2 - 9b_3) + 2\dot{b}_3(b_1^2 - 3b_2) = \\ = b_1^2 b_2^2 - 4b_1^3 b_3 + 18 b_1 b_2 b_3 - 4b_2^3 - 27 b_3^2 + \\ + 2\dot{b}_1(b_2^2 - 3b_1 b_3) - \dot{b}_2(b_1 b_2 - 9b_3) + 2\dot{b}_3(b_1^2 - 3b_2). \end{aligned}$$

Consequently, at n = 2 it is possible to apply expansion (2.40) if coefficients $b_1(t)$ and $b_2(t)$ are differentiable and the following is fulfilled:

$$b_1^2 - 4b_2 \neq 0;$$

at n = 3 expansion (2.40) will apply if coefficients $b_1(t)$, $b_2(t)$ and $b_3(t)$ are twice differentiable and the following inequalities hold:

$$\begin{aligned} 0 \neq b_1^2 b_2^2 - 4b_1^3 b_3 + 18 b_1 b_2 b_3 - 4b_2^3 - 27 b_3^2 \neq -2\dot{b}_1(b_2^2 - 3b_1 b_3) + \\ + \dot{b}_2(b_1 b_2 - 9b_3) - 2\dot{b}_3(b_1^2 - 3b_2). \end{aligned}$$

Let us note that for n = 2, conditions of applicability of first canonical expansion of second form coincide with conditions of applicability of canonical expansion of first form.

Using system (2.40) and equation (0.1), we will construct a system of equations relative to canonical components. After differentiating equations (2.40) and having compared obtained system of equations with equations of system (2.40), augmented by equation (0.1), we will find

$$\left.\begin{aligned} \dot{z}_1 + \ldots + \dot{z}_n = \lambda_1 z_1 + \ldots + \lambda_n z_n, \quad \lambda_1 \dot{z}_1 + \ldots + \lambda_n \dot{z}_n = \dot{\lambda}_1 z_1 + \ldots + \dot{\lambda}_n z_n, \\ (\dot{\lambda}_1 + \lambda_1^2)\dot{z}_1 + \ldots + (\dot{\lambda}_n + \lambda_n^2)\dot{z}_n = \lambda_1(\dot{\lambda}_1 + \lambda_1^2) z_1 + \ldots + \lambda_n(\dot{\lambda}_n + \lambda_n^2) z_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [(\lambda_1 + D)^{n-2}\lambda_1]\dot{z}_1 + \ldots [(\lambda_n + D)^{n-2}\lambda_n]\dot{z}_n = \\ = -[D(\lambda_1 + D)^{n-2}\lambda_1 + b_1(\lambda_1 + D)^{n-2}\lambda_1 + \ldots + b_{n-1}\lambda_1 + b_n] z_1 - \ldots \\ \ldots - [D(\lambda_n + D)^{n-2}\lambda_n + b_1(\lambda_n + D)^{n-2}\lambda_n + \ldots + b_{n-1}\lambda_n + b_n] z_n. \end{aligned}\right\}$$

Considering this system as a system of linear algebraic equations relative to unknowns $\dot{z}_1$, ..., $\dot{z}_n$ and solving it relative to these unknowns, we will obtain

$$\dot{z}_i = \lambda_i z_i + \sum_{j=1}^{n} h_{ij}^{(1)} z_j \qquad (2.4[illegible])$$

$$(i = 1, \ldots, n),$$

where

$$\begin{aligned} h_{ij}^{(1)} = -[(\lambda_j + D)^{n-1}\lambda_j + b_1(\lambda_j + D)^{n-2}\lambda_j + b_2(\lambda_j + D)^{n-3}\lambda_j + \ldots \\ \ldots + b_{n-1}\lambda_j + b_n] \frac{w_n^{(i)}}{W_1}; \end{aligned}$$

$w_{ni}^{(1)}$ is cofactor of element of n-th line of i-th column of matrix of coefficients of system (2.40).

In system (2.43) variables $z_1, \ldots, z_n$ and coefficients λ_i and h_{ij} ($i, j = 1, \ldots, n$) are complex-valued functions of t, with which variables $z_1, \ldots, z_n$.

a) linearly are expressed through solution x(t) and its derivatives

$$\frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}},$$

b) are differentiable functions of t, and coefficients h_{ij} satisfy condition

$$h_{ij} = \bar{h}_{kl}, \text{ if } \bar{\lambda}_i = \lambda_k \text{ and } \bar{\lambda}_j = \lambda_l.$$

Let us assume that $\zeta_1^{(1)}(t), \ldots, \zeta_n^{(1)}(t)$ are roots of characteristic equation of matrix of coefficients of right side of system (2.43)

$$\begin{Vmatrix} \lambda_1 + h_{11}^{(1)} - \zeta^{(1)} & h_{12}^{(1)} & h_{13}^{(1)} \ldots & h_{1n}^{(1)} \\ h_{21}^{(1)} & \lambda_2 + h_{22}^{(1)} - \zeta^{(1)} & h_{23}^{(1)} \ldots & h_{2n}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ h_{n1}^{(1)} & h_{n2}^{(1)} & h_{n3}^{(1)} \ldots & \lambda_n + h_{nn}^{(1)} - \zeta^{(1)} \end{Vmatrix} = 0,$$

determined as a function of parameter t. Then we will define <u>second canonical expansion</u> by system of equations (2.39), having assumed $\zeta_j(t) = \zeta_j^{(1)}(t)$ ($j = 1, \ldots, n$). Thus, system of equations determining second canonical expansion are written in the form

$$\left.\begin{aligned} x &= z_1 + \ldots + z_n, \\ \frac{dx}{dt} &= \zeta_1^{(1)} z_1 + \ldots + \zeta_n^{(1)} z_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{d^{n-1}x}{dt^{n-1}} &= [(\zeta_1^{(1)} + D)^{n-2} \zeta_1^{(1)}] z_1 + \ldots + [(\zeta_n^{(1)} + D)^{n-2} \zeta_n^{(1)}] z_n. \end{aligned}\right\} \qquad (2.44)$$

This expansion we will use only when determinant of matrix of coefficients of the right side of system (2.44) (we will designate it by symbol W_2) is different than zero during all values of t in the considered interval and coefficients during variables z_j are differentiable in the interval. In view of full analogy with conditions of applicability of first canonical expansion, it may be concluded that these conditions are fulfilled if

a) coefficients $b_1^{(1)}, \ldots, b_n^{(1)}$ of algebraic equation

$$\begin{vmatrix} \lambda_1 + h_{11}^{(1)} - \zeta^{(1)} & h_{12}^{(1)} & \ldots & h_{1n}^{(1)} \\ h_{21}^{(1)} & \lambda_2 + h_{22}^{(1)} - \zeta^{(1)} & \ldots & h_{2n}^{(1)} \\ \ldots & \ldots & \ldots & \ldots \\ h_{n1}^{(1)} & h_{n2}^{(1)} & \ldots & \lambda_n + h_{nn}^{(1)} - \zeta^{(1)} \end{vmatrix} =$$

$$= (\zeta^{(1)})^n + b_1^{(1)} (\zeta^{(1)})^{n-1} + \ldots + b_n^{(1)} = 0 \qquad (2.45)$$

are (n - 1)-fold differentiable;

b) determinant of Vandermonde relative to roots $\zeta_1^{(1)}, \ldots, \zeta_n^{(1)}$

$$W^{(1)} = \begin{vmatrix} 1 & \ldots & 1 \\ \zeta_1^{(1)} & \ldots & \zeta_n^{(1)} \\ \ldots & \ldots & \ldots \\ (\zeta_1^{(1)})^{n-1} & \ldots & (\zeta_n^{(1)})^{n-1} \end{vmatrix}$$

is different than zero;

c) $W_2 \neq 0$.

Feasibility of conditions (a), (b), and (c) it is possible to check, without determining roots of equation (2.45). Actually, as will be shown lower, coefficients $b_1^{(1)}, \ldots, b_n^{(1)}$ are rationally expressed through coefficients $b_1, \ldots, b_n$ and their derivatives from first to n - 1-st inclusively, and are limited in any finite interval if in this interval among roots $\lambda_1, \ldots, \lambda_n$ there are no multiples. Since, according to conditions of applicability of first expansion, multiplicity of roots λ_j is excluded, then coefficients $b_1^{(1)}, \ldots, b_n^{(1)}$, because of shown dependence on coefficients $b_1, \ldots, b_n$, can be n - 1-multiple differentiable in the case when coefficients $b_1, \ldots, b_n$ will be 2(n - 1)-multiple differentiable.

If condition (a) is executed, then for checking fulfillment of conditions (b) and (c) it is necessary and sufficient to check fulfillment of inequalities

$$(W^{(1)})^2 \neq 0,$$
$$W^{(1)} W_2 \neq 0.$$

Magnitudes $(W^{(1)})^2$ and $W^{(1)}W_2$ it is possible rationally to express through coefficients of equation (2.45). The latter, in turn, are rationally expressed through coefficients $b_1, \ldots, b_n$ and their derivatives of the first and highest orders. Therefore, recorded inequalities will determine the requirement that certain fractional-rational functions from coefficients $b_1, \ldots, b_n$ and their derivatives do not turn into zero when t belongs to the considered interval. Because of formulas for coefficients $h_{ij}^{(1)}$ the denominators of these fractional-rational functions contain whole degrees of magnitudes W and W_1. Since, according to conditions of applicability of first canonical expansion $W \neq 0$ and $W_1 \neq 0$, the mentioned requirement leads to inequalities which must be satisfied by certain polynomials from coefficients $b_1, \ldots, b_n$ and their derivatives.

In particular, at $n = 2$ $W^{(1)} = W_2$ and for execution of conditions (b) and (c)

it is necessary and sufficient to fulfill inequality[1]

$$(\dot{b}_1^2 - 4\dot{b}_2)^3 + 2\dot{b}_1(\dot{b}_1^2 - 4\dot{b}_2)^2 + (b_1\dot{b}_1 - 2\dot{b}_2)^2 \neq 0.$$

We will return to the determination of second canonical expansion. Let us note that in it are used functions $\zeta_i(t)$ ($i = 1, \ldots, n$), obtained as a result of certain operations from system of equations relative to canonical components of first expansion. Consequently, we approach construction of second canonical expansion only if first expansion is constructed. There exist, however, cases when conditions of applicability of first canonical expansion are not executed, and second expansion it is possible to construct. Therefore, it is expedient, having left without change the method of finding functions $\zeta_1(t), \ldots, \zeta_n(t)$, to determine conditions of applicability of second canonical expansion so that it is possible to use them independently of whether the first expansion is applicable or not.

This requirement corresponds to the above-mentioned (in initial formulation) conditions of applicability of second expansion and also conditions $(W^{(1)})^2 \neq 0$ and $W^{(1)}W_2 \neq 0$, which replaced, later, conditions (b) and (c). Regarding, however, conclusion concerning possibility of replacing condition of n - 1-multiple differentiability of coefficients $b_1^{(1)}, \ldots, b_n^{(1)}$ by condition 2(n - 1)-multiple differentiability of coefficients $b_1, \ldots, b_n$, then it is necessary to reject it as obtained during the use of conditions of applicability of the first expansion. Thus, condition (a) one should leave in initial form.

In accordance with that presented we will define conditions of applicability of second canonical expansion in the following form:

a) coefficients $b_1^{(1)}, \ldots, b_n^{(1)}$ of equation (2.45) n - 1-multiple are differentiable; b) $(W^{(1)})^2 \neq 0$; c) $W^{(1)}W_2 \neq 0$.

We will clarify cases in which first canonical expansion is inapplicable and second is applicable.

Obviously, for application of second canonical expansion there is not required fulfillment of condition $WW_1 \neq 0$. Therefore, there are possible cases when this condition is not executed and, consequently, first canonical expansion is inapplicable, and second canonical expansion it is possible to construct.

Another possibility is connected with cases in which, in all the considered interval, there is executed equality $W = 0$, and coefficients $b_1, \ldots, b_n$ are such that coefficient $b_1^{(1)}, \ldots, b_n^{(1)}$ and magnitudes $(W^{(1)})^2$ and $W^{(1)}W_2$ degenerate into

[1]This inequality is obtained from condition $(W^{(1)})^2 = (b_1^{(1)})^2 - 4b_2^{(1)} \neq 0$ with the help of below-mentioned formulas (2.52).

indeterminate forms. In a number of cases, introducing additional parameters and using passage to the limit, it is possible to evaluate these indeterminate forms. It can happen that conditions of applicability of second canonical expansion will be carried out.

Using system (2.44) and equation (0.1), we will construct a system of equations relative to canonical components. After differentiating equations (2.44) and having compared obtained system of equations with equations (2.44) and (0.1), we will find

$$\left.\begin{aligned}
&\dot z_1 + \ldots + \dot z_n = \zeta_1^{(1)} z_1 + \ldots + \zeta_n^{(1)} z_n,\\
&\zeta_1^{(1)} \dot z_1 + \ldots + \zeta_n^{(1)} \dot z_n = (\zeta_1^{(1)})^2 z_1 + \ldots + (\zeta_n^{(1)})^2 z_n,\\
&[\dot\zeta_1^{(1)} + (\zeta_1^{(1)})^2]\, \dot z_1 + \ldots + [\dot\zeta_n^{(1)} + (\zeta_n^{(1)})^2]\, \dot z_n =\\
&= \zeta_1^{(1)} [\dot\zeta_1^{(1)} + (\zeta_1^{(1)})^2]\, z_1 + \ldots + \zeta_n^{(1)} [\dot\zeta_n^{(1)} + (\zeta_n^{(1)})^2]\, z_n,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&[(\zeta_1^{(1)} + D)^{n-2} \zeta_1^{(1)}]\, \dot z_1 + \ldots + [(\zeta_n^{(1)} + D)^{n-2} \zeta_n^{(1)}]\, \dot z_n =\\
&= -[D(\zeta_1^{(1)} + D)^{n-2} \zeta_1^{(1)} + b_1 (\zeta_1^{(1)} + D)^{n-2} \zeta_1^{(1)} + \ldots\\
&\ldots + b_{n-1} \zeta_1^{(1)} + b_n]\, z_1 - \ldots - [D(\zeta_n^{(1)} + D)^{n-2} \zeta_n^{(1)} +\\
&+ b_1 (\zeta_n^{(1)} + D)^{n-2} \zeta_n^{(1)} + \ldots + b_{n-1} \zeta_n^{(1)} + b_n]\, z_n.
\end{aligned}\right| \qquad (2.46)$$

Considering system (2.46) as a system of linear algebraic equations relative to unknowns $\dot z_1, \ldots, \dot z_n$ and solving it relative to these unknowns, we will obtain

$$\dot z_i = \zeta_i^{(1)} z_i + \sum_{j=1}^{n} h_{ij}^{(2)} z_j \quad (i = 1, \ldots, n), \qquad (2.47)$$

where

$$h_{ij}^{(2)} = -[(\zeta_j^{(1)} + D)^{n-1} \zeta_j^{(1)} + b_1 (\zeta_j^{(1)} + D)^{n-2} \zeta_j^{(1)} + b_2 (\zeta_j^{(1)} + D)^{n-3} \zeta_j^{(1)} + \ldots + b_{n-1} \zeta_j^{(1)} + b_n]\, \frac{w_{ni}^{(2)}}{W_2};$$

$w_{ni}^{(2)}$ is the cofactor of the element of the n-th line of the i-th column of the matrix of coefficients of the right side of system (2.44).

In system (2.47) variables $z_1, \ldots, z_n$ and coefficients $\zeta_i^{(1)}$ and $h_{ij}^{(2)}$ (i, j = 1, ..., n) are complex-valued functions of t; variables $z_1, \ldots, z_n$ and coefficients $h_{ij}^{(2)}$ possess the same above-indicated properties, which are possessed by corresponding magnitudes of system (2.43).

Absolutely analogously it is possible to construct third and subsequent canonical expansions of the solution of an equation of oscillations. System of equations determining k + 1-st canonical expansion we will determine by a system obtained from system (2.45) by means of replacing magnitudes $\zeta_1^{(1)}, \ldots, \zeta_n^{(1)}$ by magnitudes $\zeta_1^{(k)}, \ldots, \zeta_n^{(k)}$ which are characteristic numbers of matrix of coefficients of the

right side of system

$$\dot{z}_i = \zeta_i^{(k-1)} z_i + \sum_{j=1}^{n} h_{ij}^{(k)} z_j \quad (i=1,\ldots,n), \tag{2.48}$$

which, in turn, is constructed according to system of equations determing k-th canonical expansion. With this k + 1st canonical expansion will apply only in the case when determinant $W_{\lambda+1}$, conforming to matrix of coefficients of system determining k + 1st canonical expansion, does not turn into zeroes, and coefficients during variables z_j in this system are differentiable during all values of t from the considered interval. Feasibility of these conditions it is possible to check just as this is done for second canonical expansion, without determining roots of corresponding characteristic equations. With this goal, while investigating k-th canonical expansion, it is necessary, first, to be convinced of (n - 1)-multiple differentiability of coefficients $b_1^{(k)}, \ldots, b_n^{(k)}$ of equation determining magnitudes $\zeta_1^{(k)}, \ldots, \zeta_n^{(k)}$, and, secondly, to investigate determinant W_{k+1} and determinant of Vandermonde about roots $\zeta_1^{(k)}, \ldots, \zeta_n^{(k)} W^{(k)}$. Product $W_{k+1}W^{(k)}$ will be rationally expressed through coefficients and derivatives of coefficients of characteristic equation, corresponding to matrix of the coefficients of the system determining the preceding k-th canonical expansion; coefficients of each subsequent characteristic equation are rationally expressed through coefficients and products of coefficients of preceding equation (see lower). As a result, product $W_{k+1}W^{(k)}$ will be rationally expressed through coefficients $b_1(t), \ldots, b_n(t)$ and their derivatives, first and highest. Magnitude $(W^{(k)})^2$ is rationally expressed through coefficients $b_1^{(k)}, \ldots, b_n^{(k)}$. If inequalities

$$(W^{(k)})^2 \neq 0, \quad W^{(k)} W_{k+1} \neq 0$$

are fulfilled, then determinants W_{k+1} and $W^{(k)}$ do not take zero values. Fulfillment of these inequalities jointly with condition of (n - 1)-multiple differentiability of coefficients $b_1^{(k)}, \ldots, b_n^{(k)}$ is sufficiently for above-mentioned conditions to be fulfilled.

System of equations relative to canonical components for k + 1st expansion we will construct using system of equations determining this expansion and equation (0.1) (just as this is done for second canonical expansion), and we will give to it the form

$$\dot{z}_i = \zeta_i^{(k)} z_i + \sum_{k=1}^{n} h_{ij}^{(k-1)} z_j \quad (i=1,\ldots,n). \tag{2.49}$$

where

$$h_{ij}^{(k+1)} = -[(\zeta_j^{(k)}+D)^{n-1}\zeta_j^{(k)} + b_1(\zeta_j^{(k)}+D)^{n-2}\zeta_j^{(k)} + b_2(\zeta_j^{(k)}+D)^{n-3}\zeta_j^{(k)} + \ldots + b_{n-1}\zeta_j^{(k)} + b_n]\frac{w_{ni}^{(k+1)}}{W_{k+1}};$$

$w_{ni}^{(k+1)}$ is the cofactor of the element of the n-th line of the i-th column of the matrix of coefficients of the system determining k-th canonical expansion.

In system (2.49) canonical components $z_1, \ldots, z_n$ and coefficients $\zeta_i^{(k)}$ and $h_{ij}^{(k+1}$ (j, i = 1, ..., n), as in earlier considered cases, are complex-values functions of t; variables $z_1, \ldots, z_n$ also are differentiable functions of t and are linearly expressed through solution x(t) and its derivatives dx/dt, ..., $d^{n-1}x/dt^{n-1}$ and coefficients $h_{ij}^{(k+1)}$ possess properties $h_{ij}^{(k+1)} = \bar{h}_{pq}^{(k+1)}$ if $\lambda_i = \bar{\lambda}_p$, $\lambda_j = \bar{\lambda}_q$.

Fi d recurrence relationship between constructed canonical expansion determines the sequence of canonical expansions. This sequence, depending upon form of investigated equation, can have finite or infinite number of members.

An important circumstance facilitating construction of canonical expansions of the highest orders consists in the fact that for construction of k + 1st canonical expansion there is no necessity for calculation of roots λ_j, $\zeta_j^{(1)}, \ldots, \zeta_j^{(k-1)}$ (j = 1, ..., n) determining the preceding canonical expansions. This follows from the fact that coefficients of the characteristic equation connected with k + 1st canonical expansion are rationally expressed through coefficients and derivatives of coefficients of characteristic equation connected with the canonical expansion, i.e., in the end, through coefficients and derivatives (first and highest) of coefficients of equation of free oscillations.

In order to establish dependence between coefficients of characteristic equations

$$(\zeta^{(k)})^n + b_1^{(k)}(\zeta^{(k)})^{n-1} + \ldots + b_n^{(k)}\zeta^{(k)} = 0$$

and

$$(\zeta^{(k-1)})^n + b_1^{(k-1)}(\zeta^{(k-1)})^{n-1} + \ldots + b_n^{(k-1)}\zeta^{(k-1)} = 0,$$

the roots of which determine k + 1st and k-th canonical expansions, we will turn to system (2.48) and, in accordance with form of matrix of its coefficients, will write tne first of the shown characteristic equations in the form

$$\begin{vmatrix} \zeta_1^{(k-1)}+h_{11}^{(k)}-\zeta^{(k)} & h_{12}^{(k)},\ldots, h_{1n}^{(k)} \\ h_{21}^{(k)} & \zeta_2^{(k-1)}+h_{22}^{(k-1)}-\zeta^{(k)},\ldots, h_{2n}^{(k)} \\ \cdots & \cdots \\ h_{n1}^{(k)} & h_{n2}^{(k)},\ldots \zeta_n^{(k-1)}+h_{nn}^{(k)}-\zeta^{(k)} \end{vmatrix}=0$$

Evaluating determinant, we will find

$$\left.\begin{aligned} b_1^{(k)} &= -(\zeta_1^{(k-1)}+\ldots+\zeta_n^{(k-1)}+h_{11}^{(k)}+\ldots+h_{nn}^{(k)}) = \\ &= b_1^{(k-1)}-H_1^{(k)}, \\ b_2^{(k)} &= \zeta_1^{(k-1)}\zeta_2^{(k-1)}+\ldots+\zeta_{n-1}^{(k-1)}\zeta_n^{(k-1)}+ \\ &+h_{11}^{(k)}(-b_1^{(k-1)}-\zeta_1^{(k-1)})+\ldots+h_{nn}(-b_n^{(k-1)}-\zeta_n^{(k-1)}) = \\ &= b_2^{(k-1)}-b_1^{(k-1)}H_1^{(k)}-H_2^{(k)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ b_l^{(k)} &= b_l^{(k-1)}-b_{l-1}^{(k-1)}H_1^{(k)}-b_{l-2}^{(k-1)}H_2^{(k)}-\ldots-H_l^{(k)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ b_n^{(k)} &= b_n^{(k-1)}-b_{n-1}^{(k-1)}H_1^{(k)}-b_{n-2}^{(k-1)}H_2^{(k)}-\ldots-H_n^{(k)}, \end{aligned}\right\} \quad (2.50)$$

where

$$H_l^{(k)} = (\zeta_1^{(k-1)})^{l-1}h_{11}^{(k)}+\ldots+(\zeta_n^{(k-1)})^{l-1}h_{nn}^{(k)}$$
$$(l=1,\ldots,n).$$

By the formulas mentioned above, coefficients $h_{ij}^{(k)}$ are rationally expressed through coefficients b_1, ..., b_n, roots $\zeta^{(k-1)}$, ..., $\zeta_n^{(k-1)}$ and derivatives of these roots from the first to n - 1st, inclusively. These depedences can be converted after ridding them of derivatives of roots with the help of Viete's formulas.

Actually, from Viete's formulas follows

$$\left.\begin{aligned} &\dot\zeta_1^{(k-1)}+\ldots+\dot\zeta_n^{(k-1)} = -\dot b_1^{(k-1)}, \\ &(b_1^{(k-1)}+\zeta_1^{(k-1)})\dot\zeta_1^{(k-1)}+\ldots+(b_1^{(k-1)}+\zeta_n^{(k-1)})\dot\zeta_n^{(k-1)} = -\dot b_2^{(k-1)}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &[b_{n-1}^{(k-1)}+\zeta_1^{(k-1)}b_{n-2}^{(k-1)}+\ldots+(\zeta_1^{(k-1)})^{n-1}]\dot\zeta_1^{(k-1)}+\ldots+ \\ &+[b_{n-1}^{(k-1)}+\zeta_n^{(k-1)}b_{n-2}^{(k-1)}+\ldots+(\zeta_n^{(k-1)})^{n-1}]\dot\zeta_n = -\dot b_n^{(k-1)}. \end{aligned}\right| \quad (2.51)$$

It is not difficult to be convinced that determinant of system (2.51) is determinant of Vandermonde $W^{(k-1)}$. Really, after presenting second line of this determinant in the form of the sum of

$$(b_1^{(k-1)},\ldots,b_n^{(k-1)})+(\zeta_1^{(k-1)},\ldots,\zeta_n^{(k-1)}),$$

we obtain two component determinants. The first of them is equal to zero, since the first and second lines of it are proportional; consequently, determinant of system will coincide with second component determinant. Presenting this latter in

the form of the sum of three determinants with different third lines

$$(b_2^{(k-1)}, \dots, b_2^{(k-1)}),$$
$$(b_1^{(k-1)}\zeta_1^{(k-1)}, \dots, b_1^{(k-1)}\zeta_n^{(k-1)}),$$
$$|(\zeta_1^{(k-1)})^2, \dots (\zeta_n^{(k-1)})^2|,$$

we find that the first two turn into zeroes, as determinants with proportional lines, etc. Continuing the process, we will find that the remainder not turning into zeroes coincides with determinant of Vandermonde.

If the following inequality holds

$$W^{(k-1)} \neq 0,$$

then system (2.51) may be solved relative to magnitudes $\dot{\zeta}_1^{(k-1)}, \dots, \dot{\zeta}_n^{(k-1)}$ and, consequently, derivatives of roots can be expressed through roots, coefficients $b_1^{(k-1)}, \dots, b_n^{(k-1)}$, and first derivatives of the latter.

After obtaining corresponding formulas, differentiating them term by term, and repeatedly applying them for elimination in right sides of derivatives of roots, we will obtain formulas expressing second derivatives of roots through roots, coefficients $b_1^{(k-1)}, \dots, b_n^{(k-1)}$, first and second derivatives of the latter.

It is possible to proceed analoguusly with highest derivatives of roots, entering into expression for coefficients $h_{jj}^{(k)}$. After executing all these transformations, we will be convinced that functions $H_l^{(k)}$ are symmetric functions of roots with coefficients rationally depending on coefficients $b_1^{(k-1)}, \dots, b_n^{(k-1)}$ and their derivatives, from first to n - 1st, inclusively. This allows us to express coefficients $b_1^{(k)}, \dots, b_n^{(k)}$ in the form of fractional, rational functions of coefficients $b_1^{(k-1)}, \dots, b_n^{(k-1)}$ and their above-mentioned derivatives.

In particular, at n = 2 relationships between coefficients of algebraic equations $b_1^{(k)}, \dots, b_n^{(k)}$ and $b_1^{(k-1)}, \dots, b_n^{(k-1)}$, connected with k + 1st and k-th canonical expansions have the form

$$\left.\begin{aligned} b_1^{(k)} &= b_1 + \frac{b_1^{(k-1)}\dot{b}_1^{(k-1)} - 2\dot{b}_2^{(k-1)}}{(b_1^{(k-1)})^2 - 4b_2^{(k-1)}}, \\ b_2^{(k)} &= b_2 + \frac{2\dot{b}_1^{(k-1)}b_2^{(k-1)} - b_1^{(k-1)}\dot{b}_2^{(k-1)}}{(b_1^{(k-1)})^2 - 4b_2^{(k-1)}} \qquad (k = 1, 2, \dots). \end{aligned}\right| \qquad (2.[illegible])$$

Above it was noticed that canonical components can be both real and complex functions of t even if solution x(t) is real. For real solutions x(t) between canonical components $z_1, \dots, z_n$ and roots determining given canonical expansion, the same relationship occurs as between canonical components $y_1, \dots, y_n$ and roots $\lambda_1, \dots, \lambda_n$: canonical component z_j is real is corresponding root is complex;

conjugate roots correspond to conjugate canonical components.

During construction of considered sequence of canonical expansions of solution of equation of oscillations, there are possible cases when l-th canonical expansion changes equation of oscillations into a system whose matrix of coefficients is diagonal. Every i-th equation of such a system constitutes a linear uniform differential equation relative to variable z_1, solution of which is also solution of equation of oscillations.

For clarification of whether the matrix of coefficients of any system of equations relative to canonical components is diagonal, there is no necessity for determining coefficients of this matrix; because of formulas of transition from coefficients of algebraic equation determining parameters of k-th canonical expansion to analogous coefficients of equation determining parameters of k + 1st expansion, matrix of coefficients of system of equations, obtained as a result of k-th canonical expansion, is diagonal if and only if

$$b_j^{(k)} = b_j^{(k-1)} \qquad j = (1, \ldots, n).$$

Example. For equation of Fuler

$$\frac{d^2x}{dt^2} + \frac{c}{t^2}x = 0, \tag{2.53}$$

as in general for any second order equation, first canonical expansion of the form (2.39) coincides with canonical expansion considered in preceding paragraph.

By the formulas (2.52) for coefficients of algebraic equation, from which are determined functions $\zeta_1^{(1)}(t)$ and $\zeta_2^{(1)}(t)$, we obtain

$$b_1^{(1)} = -\frac{1}{t},$$

$$b_2^{(1)} = \frac{c}{t^2}.$$

It follows from this that

$$\zeta_{1,2}^{(1)} = \frac{1}{2t} \mp \frac{1}{2t}\sqrt{1-4c}.$$

First canonical expansion for considered equation is inapplicable if c = 0. Second canonical expansion is inapplicable if 1 - 4c = 0. Excluding the latter case, we will obtain system of equations determining second canonical expansion for t > 0 in the form

$$\left.\begin{aligned} x &= z_1 + z_2, \\ \frac{dx}{dt} &= +\frac{1}{2t}\left(1-\sqrt{1-4c}\right)z_1 + \frac{1}{2t}\left(1+\sqrt{1-4c}\right)z_2. \end{aligned}\right\}$$

In accordance with equations (2.47) we will constitute system of equations relative to canonical components. With this goal, preliminarily we will calculate coefficients $h_{i,j}^{(2)}$:

$$W_2 = \frac{\sqrt{1-4c}}{t}; \quad \alpha_{21}^{(2)} = -1, \quad \alpha_{22}^{(2)} = 1;$$

$$(\zeta_1^{(1)})^2 + \dot{\zeta}_1^{(1)} + \frac{c}{t^2} = 0; \quad (\zeta_2^{(1)})^2 + \dot{\zeta}_2^{(1)} + \frac{c}{t^2} = 0,$$

whence

$$h_{11}^{(2)} = h_{12}^{(2)} = h_{21}^{(2)} = h_{22}^{(2)} = 0.$$

Because of the last equality, system of equations relative to canonical components takes the form

$$\left.\begin{aligned} \dot{z}_1 &= \frac{1}{2t}\left(1-\sqrt{1-4c}\right) z_1, \\ \dot{z}_2 &= \frac{1}{2t}\left(1+\sqrt{1+4c}\right) z_2, \end{aligned}\right|$$

i.e., is broken down into two independent equations, each of which is simply integrated. After finding solutions and summing them, we will obtain a general solution of the investigated equation (see example in § 3).

Thus, in the considered example, application of the second canonical expansion has allowed us to find exact solution of equation of free oscillations. This case, of course, is exceptional. However, it indicates the wide possibilities which the application of the construction of canonical expansions given in this paragraph conceals.

§ 5. Canonical Expansions of Solution of Equations of Oscillations During Disturbance of Condition Nonmultiplicity of Roots $\lambda_i(t)$ or $\zeta_i^{(k)}(t)$

Till now we have only considered such canonical expansions of the solution of an equation of free oscillations, whose region of application is limited by the requirement that during all values of t from the interval interesting us all functions $\lambda_i(t)$ or $\zeta_i^{(k)}(t)$ had different values. This section is devoted to cases in which this requirement is not fulfilled.

Excluding such specific cases when in the interval interesting us equal values take two or more pairs of functions $\lambda_i(t)$ or $\zeta_i^{(k)}(t)$, we will assume that coincidence of values of functions $\lambda_i(t)$ or $\zeta_i^{(k)}(t)$ takes place only for one pair of these functions. Without disturbing generality, we will ascribe to these functions indices "n - 1" and "n."

Introducing functions $\hat{\lambda}_{n-1}(t)$ and $\hat{\lambda}_n(t)$ or $\hat{\zeta}_{n-1}^{(k)}(t)$ and $\hat{\zeta}_n^{(k)}(t)$ by conditions

$$\hat{\lambda}_{n-1}(t) = \frac{\lambda_{n-1}(t) + \lambda_n(t)}{2}, \quad \hat{\lambda}_n(t) = 0. \qquad (2.54)$$

$$\hat{\zeta}_{n-1}^{(k)}(t) = \frac{\zeta_{n-1}(t) + \zeta_n(t)}{2}, \quad \hat{\zeta}_n^{(k)}(t) = 0. \qquad (2.55)$$

we will replace in the system of equations of canonical expansion (2.27) functions $\lambda_{n-1}(t)$ and $\lambda_n(t)$ by functions $\hat{\lambda}_{n-1}(t)$ and $\hat{\lambda}_n(t)$, and in the system of equations of $(k+1)$-st canonical expansion of the form (2.39) we will replace functions $\zeta_{n-1}^{(k)}$ and $\zeta_n^{(k)}(t)$ by functions $\hat{\zeta}_{n-1}^{(k)}$ and $\hat{\zeta}_n^{(k)}(t)$. We will obtain, instead of system (2.27), system

$$\left.\begin{aligned}
&x = y_1 + \ldots + y_{n-2} + y_{n-1} + y_n,\\
&\frac{dx}{dt} = \lambda_1 y_1 + \ldots + \lambda_{n-2} y_{n-2} + \frac{\lambda_{n-1} + \lambda_n}{2}\, y_{n-1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n-1}x}{dt^{n-1}} = \lambda_1^{n-1} y_1 + \ldots + \lambda_{n-2}^{n-1} y_{n-1} + \left(\frac{\lambda_{n-1} + \lambda_n}{2}\right)^{n-1} y_{n-1}.
\end{aligned}\right| \qquad (2.56)$$

and, instead of system of the form (2.39), system

$$\left.\begin{aligned}
&x = z_1 + \ldots + z_{n-2} + z_{n-1} + z_n,\\
&\frac{dx}{dt} = \zeta_1^{(k)} z_1 + \ldots + \zeta_{n-2}^{(k)} z_{n-2} + \frac{\zeta_{n-1}^{(k)} + \zeta_n^{(k)}}{2}\, z_{n-1},\\
&\frac{d^2x}{dt^2} = [(\zeta_1^{(k)} + D)\,\zeta_1^{(k)}]\, z_1 + \ldots + [(\zeta_{n-2}^{(k)} + D)\,\zeta_{n-2}^{(k)}]\, z_{n-2} +\\
&\qquad + \left[\left(\frac{\zeta_{n-1}^{(k)} + \zeta_n^{(k)}}{2} + D\right)\frac{\zeta_{n-1}^{(k)} + \zeta_n^{(k)}}{2}\right] z_{n-1},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n-1}x}{dt^{n-1}} = [(\zeta_1^{(k)} + D)^{n-2}\,\zeta_1^{(k)}]\, z_1 + \ldots + [(\zeta_{n-2}^{(k)} + D)^{n-2}\,\zeta_{n-2}^{(k)}]\, z_{n-2} +\\
&\qquad + \left[\left(\frac{\zeta_{n-1}^{(k)} + \zeta_n^{(k)}}{2} + D\right)^{n-2}\frac{\zeta_{n-1}^{(k)} + \zeta_n^{(k)}}{2}\right] z_{n-1}.
\end{aligned}\right| \qquad (2.57)$$

Determinant of matrix of coefficients of the right side of system (2.56) is equal to magnitude

$$\hat{W} = (-1)^n \lambda_1 \ldots \lambda_{n-2} \hat{\lambda}_{n-1} W(\lambda_1, \ldots, \lambda_{n-2}, \hat{\lambda}_{n-1}), \qquad (2.58)$$

where $W(\lambda_1, \ldots, \lambda_{n-1}, \hat{\lambda}_{n-1})$ is determinant of Vandermonde for magnitudes $\lambda_1, \ldots, \lambda_{n-1}, \hat{\lambda}_{n-1}$.

Determinant $W(\lambda_1, \ldots, \lambda_{n-1}, \hat{\lambda}_{n-1})$, because of the expressed-above assumption about multiplicity of only one pair of roots, is different than zero. Therefore, magnitude $\hat{W}$ is different than zero in all cases besides the case when one of the magnitudes $\lambda_1, \ldots, \lambda_{n-1}, \hat{\lambda}_{n-1}$ is equal to zero.

We will assume that all magnitudes $\lambda_1, \ldots, \lambda_{n-1}, \hat{\lambda}_{n-1}$ are different than zero. Taking into account this limitation, it is possible to derive from system of equations (2.56) and equation (0.1) a system of equations which canonical components $y_1, \ldots, y_n$ satisfy. Obviously, it may be obtained from system (2.43), if one were to produce

in it replacement

$$\zeta_{n-1}(t)\to\hat{\zeta}_{n-1}(t),\quad \zeta_n(t)\to\hat{\zeta}_n(t),\quad z_i\to y_i\quad (i=1,\ldots,n).$$

Determinant of matrix of coefficients of the right side of system (2.57) is equal to magnitude

$$\hat{W}_{k+1}=(-1)^n\times$$
$$\times\begin{vmatrix} \zeta_1^{(k)} & \ldots & \zeta_{n-2}^{(k)} & \dfrac{\zeta_{n-1}^{(k)}+\zeta_n^{(k)}}{2} \\ [(\zeta_1^{(k)}+D)\zeta_1^{(k)}] & \ldots & [(\zeta_{n-2}^{(k)}+D)\zeta_{n-2}^{(k)}] & \left[\left(\dfrac{\zeta_{n-1}^{(k)}+\zeta_n^{(k)}}{2}+D\right)\dfrac{\zeta_{n-1}^{(k)}+\zeta_n^{(k)}}{2}\right] \\ \ldots & \ldots & \ldots & \ldots \\ [(\zeta_1^{(k)}+D)^{n-2}\zeta_1^{(k)}] & \ldots & [(\zeta_{n-2}^{(k)}+D)^{n-2}\zeta_{n-2}^{(k)}] & \left[\left(\dfrac{\zeta_{n-1}^{(k)}+\zeta_n^{(k)}}{2}+D\right)^{n-2}\dfrac{\zeta_{n-1}^{(k)}+\zeta_n^{(k)}}{2}\right] \end{vmatrix} \tag{2.59}$$

. Being limited by cases in which for all t magnitude $\hat{W}_{k+1}\neq 0$, we will derive from system of equations (2.57) and equation (0.1) a system of equations relative to canonical components $z_1, \ldots, z_n$. Obviously, this system can be obtained from system of equations (2.49), if the latter we replace variable $\zeta_{n-1}^{(k)}(t)$ by variable $\hat{\zeta}_{n-1}^{(k)}(t)$ and variable $\zeta_n^{(k)}(t)$ by variable $\hat{\zeta}_n^{(k)}(t)$.

Canonical components, obtained as a result of the above-indicated modified canonical expansions, possess the same properties which canonical components of corresponding unmodified canonical expansions possess (see §§ 3 and 4).

In order to explain the meaning of the presented modifications of structures of expansions, we will assume that $\zeta_{n-1}^{(k)}(t) = \zeta_n^{(k)}(t)$ and that functions

$$C_i \exp \zeta_i^{(k)}(t) \qquad (i=1,\ldots,n)$$

are solutions of equation (0.1). Then system of equations relative to canonical component for k + 1st expansion will have the form

$$\left.\begin{aligned} \dot{z}_1 &= \zeta_1^{(k)} z_1 + h_{1n}^{(k+1)} z_n, \\ \dot{z}_2 &= \zeta_2^{(k)} z_2 + h_{2n}^{(k+1)} z_n, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \dot{z}_{n-1} &= \hat{\zeta}_{n-1}^{(k)} z_{n-1} + h_{n-1,n}^{(k+1)} z_n, \\ \dot{z}_n &= h_{nn}^{(k+1)} z_n. \end{aligned}\right| \tag{2.60}$$

Since matrix of system (2.60) is triangular, this system may be solved by means of consecutive determination of unknowns. In this case, the last equations of the system may be used for determination of solution with respect to variable $z_n(t)$. Solution with respect to another variable is obtained by means of separate integration of the other equations, in which unknown z_n is replaced by the already found solution with respect to this variable.

From that presented it may be concluded that modified structure of canonical expansion of solution of equation (0.1) to components $z_1, \ldots, z_n$ allows the possibility for the matrix of the system of equations relative to canonical components to have triangular form, which allows this system to be integrated into quadratures. It is possible to expect that for a certain class of equations (0.1) such structure of canonical expansion is reduced to a system of equations relative to canonical components, which is not integrated in quadratures, but is in some meaning, close to a system with a matrix of coefficients of triangular form.

Modified canonical expansions of solution of equation (0.1) to components $y_1, \ldots, y_n$ is not due to the modifications but is a general deficiency of canonical expansions of this form. Such expansions, however, have advantages connected with the comparative simplicity of their practical application.

For illustration of the method of construction of systems of equations relative to canonical component during application of the considered modified canonical expansions, we will consider an example.

<u>Example</u>. Equation of Euler

$$\frac{d^2x}{dt^2} + \frac{c}{t^2}x = 0$$

at $c = 1/4$ does not allow second canonical expansion of the unmodified structure, since in this case

$$\zeta_1^{(1)}(t) = \zeta_2^{(1)}(t) = \frac{1}{2t}$$

(see example in § 4). We will construct a modified canonical expansion corresponding to these roots.

We have

$$\hat{\zeta}_1^{(1)}(t) = \frac{1}{2t}, \quad \hat{\zeta}_2^{(1)}(t) = 0.$$

In accordance with formulas for coefficients $h_{ij}^{(2)}$ of equation (2.47) we will obtain these coefficients in the form

$$h_{11}^{(2)} = h_{21}^{(2)} = 0, \quad h_{12}^{(2)} = -h_{22}^{(2)} = -\frac{1}{2t}.$$

System of equations relative to canonical components will take the form

$$\left.\begin{aligned} \dot{z}_1 &= \frac{1}{2t} z_1 - \frac{1}{2t} z_2, \\ \dot{z}_2 &= \frac{1}{2t} z_2. \end{aligned}\right|$$

From second equations we will find

$$z_2 = C_1 \exp\frac{1}{2}\ln t = C_1\sqrt{t},$$

where C_i is arbitrary constant.

Substituting found solution for variable z_2 in the first equation, we will obtain

$$\dot{z}_1 = \frac{1}{2t} z_1 - \frac{C_1}{2\sqrt{t}}.$$

By the formula of solution of an equation of the first order we will find

$$z_1 = \sqrt{t}\left(C_2 - \frac{1}{2C_1} \ln t\right).$$

Summarizing z_1 and z_2, we will obtain known formula of general solution of equation of Euler for considered case:

$$x = z_1 + z_2 = \sqrt{t}\,(C_1' + C_2' \ln t).$$

where C_1' and C_2' are arbitrary constants.

CHAPTER III

ELEMENTS OF THEORY OF SYSTEMS OF LINEAR UNIFORM DIFFERENTIAL EQUATIONS

§ 1. Class of Considered Systems

Considered in the preceding chapter, canonical expansions of the solution of an equation of free oscillations allow us to set up in conformity to this equation, certain systems of linear uniform first order differential equations, solved relative to derivatives whose coefficients are continuous, differentiable, and, in general, complex. In this chapter are expounded certain elements of a theory of systems possessing the shown properties.

Thus, let us assume that system interesting us is recorded in the form

$$\dot{x}_i = a_{i1}x_1 + \ldots + a_{in}x_n \quad (i = 1, \ldots, n), \tag{3.1}$$

where coefficients a_{ij} are continuous, differentiable, and, in general, complex-valued functions of real variable t, definite in region $[\alpha, \beta)$, where $\alpha \geq 0$ and $\beta \leq \infty$.

§ 2. Questions of Existence and Singleness of Solutions

It is known that if coefficients α_{ij} $(i, j = 1, \ldots, n)$ are continuous in half-open interval $[\alpha, \beta]$, then system of equations (3.1) has in this interval only particular solution, satisfying given initial conditions

$$x_i(\alpha) = \xi_i \qquad (i = 1, \ldots, n). \tag{3.2}$$

where ξ_i $(i = 1, \ldots, n)$ are given numbers. With this, functions $x_i(t)$ are limited in any finite interval belonging to interval (α, β) [22-24] and are differentiable in interval (α, β).

Of interest is one more property of system of equations (3.1). The presence of this property establishes the following theorem of Wintner.

Wintner Theorem [29]. If all coefficients of a system of differential equations, recorded in the form of (3.1), are continuous and absolutely integrable in certain finite interval (α, β), then all elements of fundamental system of its solutions are limited during $t = \alpha$.

Because of assumed (in § 1) properties of coefficients a_{ij} $(i, j = 1, \ldots, n)$ the latter satisfy conditions of the given theorem. Applying this theorem and considering the relationship between fundamental system of solutions and general solution (see § 6 Chapter 1), we find that system (3.1) has only such particular solutions at which function $x_i(t)$ $(i = 1, \ldots, n)$ take finite values at $t = \alpha$.

§ 3. Norm and Phase Coefficients of Solutions

We will agree to call the following magnitude norm of solution of system (3.1)

$$r(t) = \sqrt{x_1\bar{x}_1 + \ldots + x_n\bar{x}_n} \tag{3.3}$$

and phase coefficients of solution will be magnitudes $e_1(t), \ldots, e_n(t)$, connected with variables $x_1, \ldots, x_n$ and norm r equalities

$$x_i = re_i \qquad (i = 1, \ldots, n). \tag{3.4}$$

From equality (3.3) it is easy to conclude that norm of solution is equal to zero if and only if all variables x_i are equal to zero. Since the latter conditions are executed for zero solution, then norm of zero solution is equal to zero; norm of any other solution is always positive. If two particular solutions of system (3.1) $x_1'(t), \ldots, x_n'(t)$ and $x''(t), \ldots, x_n''(t)$ are connected by dependences

$$x_i''(t) = Cx_i'(t) \qquad (i = 1, \ldots, n), \tag{3.5}$$

where C is real constant, then the same relationship takes place between norms of solutions. Thus, norm of nontrivial solution is a certain measure of deviation of this solution from zero.

Phase coefficients characterize relationship between variables $x_1, \ldots, x_n$. Analogous phase coefficients e_i' and e_i'' $(i = 1, \ldots, n)$ in two different cases coincide, if corresponding functions $x_i'(t)$ and $x_i''(t)$ are connected by proportional dependence (3.5) with real positive proportionality factor C. By definition [see formula (3.3) and (3.4)], phase coefficients are connected by dependence

$$e_1\bar{e}_1 + \ldots + e_n\bar{e}_n = 1. \tag{3.6}$$

Let us note that functions $r(t)$ and $e_i(t)$ ($i = 1, \ldots, n$) are continuous and m-multiple differentiable if functions $x_i(t)$ ($i = 1, \ldots, n$) are continuous and m-multiple differentiable, and that these functions are analytic functions of t if functions $x_i(t)$ ($i = 1, \ldots, n$) possess the same properties.

§ 4. Connection Between Logarithmic Derivative of Norm and Phase Coefficients

From equality (3.3) follows

$$r\dot{r}=\frac{1}{2}\sum_{i=1}^{n}(\dot{x}_i\bar{x}_i+x_i\dot{\bar{x}}_i). \tag{3.7}$$

Excluding from equality (3.7) derivatives of magnitudes x_i and $\bar{x}_i$ [using equations (3.1)], we will obtain

$$\dot{r}r=\frac{1}{2}\sum_{i=1}^{n}\left(\bar{x}_i\sum_{j=1}^{n}a_{ij}x_j+x_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{x}_j\right). \tag{3.8}$$

After dividing term by term, these equations by r^2, we will find

$$\frac{\dot{r}}{r}=\frac{1}{2}\sum_{i=1}^{n}\left(\bar{e}_i\sum_{j=1}^{n}a_{ij}e_j+e_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{e}_j\right). \tag{3.9}$$

Let us note that

$$\frac{\dot{r}}{r}=\frac{d}{dt}\ln r.$$

Consequently, formula (3.9) establishes the connection between logarithmic derivative of norm of solution and its phase coefficients. This formula allows us for arbitrary value of t to determine logarithmic derivative of norm if are known values of phase coefficients with this value of t.

Example. System of equations relative to canonical components of solution of particular form of equation of Euler

$$\frac{d^2x}{dt^2}+\frac{1}{4t^2}x=0,$$

obtained as a result of second canonical expansion of modified structure, has the form

$$\left.\begin{aligned}\dot{x}_1&=\frac{1}{2t}x_1-\frac{1}{2t}x_2,\\ \dot{x}_2&=\frac{1}{2t}x_2\end{aligned}\right|$$

(see § 5 Chapter II. It is assumed: $z_{1,2} = x_{1,2}$).

Its coefficients are continuous and differentiable in interval $(0, \infty)$. Considering this interval and applying formula (3.9), we will find

$$\frac{d}{dt}\ln r=\frac{1}{4t}(2e_1\bar{e}_1-e_1\bar{e}_2-\bar{e}_1e_2+2e_2\bar{e}_2).$$

§ 5. System of Differential Equations Relative to Phase Coefficients

We will return now to equalities (3.4). They have to be executed during all values of t from interval interesting us. After differentiating both parts of these equalities, we will obtain

$$\dot{x}_i = \dot{r}e_i + r\dot{e}_i \quad (i = 1, \ldots, n). \tag{3.10}$$

Excluding from equalities (3.10) derivatives of variables x_i, on the basis of equations (3.1) we will find

$$\sum_{j=1}^{n} a_{ij}x_j = \dot{r}e_i + r\dot{e}_i \quad (i = 1, \ldots, n). \tag{3.11}$$

After term by term division of equations (3.11) by magnitude r, we will obtain

$$\dot{e}_i = \sum_{j=1}^{n} a_{ij}e_j - \frac{\dot{r}}{r}e_i \quad (i = 1, \ldots, n). \tag{3.12}$$

Excluding magnitude $\dot{r}/r$ with the help of formula (3.9) from equations (3.12), we will find

$$\dot{e}_i = \sum_{j=1}^{n} a_{ij}e_j - \frac{1}{2}e_i \sum_{i=1}^{n}\left(\bar{e}_i \sum_{j=1}^{n} a_{ij}e_j + e_i \sum_{j=1}^{n} \bar{a}_{ij}\bar{e}_j\right) \quad (i = 1, \ldots, n). \tag{3.13}$$

Equations (3.13) constitute a system of nonlinear differential equations relative to phase coefficients. From all its solutions, we are interested only in those for which at $t = t_0$ there is executed condition (3.6). Such solutions exist for every combination of initial values of phase coefficients, satisfying condition (3.6). These solutions for given initial values of coefficients are unique and satisfy condition (3.6) also during all $t > t_0$.

Example. For particular form of equations of Euler, considered in preceding paragraph, system of differential equations relative to phase coefficients has the form

$$\left.\begin{aligned} \dot{e}_1 &= \frac{1}{2t}(e_1 - e_2) - \frac{e_1}{4t}(2e_1\bar{e}_1 - e_1\bar{e}_2 - \bar{e}_1e_2 + 2e_2\bar{e}_2), \\ \dot{e}_2 &= \frac{1}{2t}e_2 - \frac{e_2}{4t}(2e_1\bar{e}_1 - e_1\bar{e}_2 - \bar{e}_1e_2 + 2e_2\bar{e}_2). \end{aligned}\right\}$$

§ 6. Elements of Theory of Quadratic and Hermitian Forms

In formula (3.9), expressing connection between logarithmic derivative of solution and phase coefficients, and in equations (3.13) forming system of equations relative to phase coefficients is contained expression

$$\frac{1}{2}\sum_{i=1}^{n}\left(\bar{e}_i\sum_{j=1}^{n}a_{ij}e_j+e_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{e}_j\right).$$

This expression, considered as function of magnitudes $e_1, \ldots, e_n$, constitutes either quadratic form [27] (if all magnitudes $e_1, \ldots, e_n$ are real), or Hermitian form [33] (if all or some of magnitudes $e_1, \ldots, e_n$ are complex). Therefore, before continuing account of theory of systems, we will consider certain known ideas and positions from theory of quadratic and Hermitian forms [33].

According to definition, a uniform polynomial of second degree relative to n variables $e_1, \ldots, e_n$ is called quadratic form. Quadratic form can always be presented in the form

$$\sum_{i,j=1}^{n}c_{ij}e_ie_j \quad (c_{ij}=c_{ji};\ i, j=1, \ldots, n), \tag{3.14}$$

where c_{ij} are elements of symmetric matrix[1]

$$C=\|c_{ij}\|_1^n.$$

If C is real symmetric matrix, then form (3.14) is called real. We will assume subsequently that matrix C is real.

Determinant

$$\det|c_{ij}|_1^n.$$

is called discriminant of quadratic form. Form is called singular if its discriminant is equal to zero.

Real quadratic form (3.14) it is possible, by an infinite number of methods, to present in the form

$$\sum_{i,j=1}^{n}c_{ij}e_ie_j=\sum_{i=1}^{r}c_iE_i^2. \tag{3.15}$$

where $r(\leq n)$ is rank of matrix C, called also rank of quadratic form (3.14), $c_i \neq 0$ $(i = 1, \ldots, r)$ are real numbers, $E_i=\sum_{j=1}^{n}\gamma_{ij}e_j\ (i=1, \ldots, n)$ are independent, real, linear combinations of variables $e_1, \ldots, e_n$.

---

[1]A symmetric matrix is one whose coefficients c_{ij} are connected by the dependence $c_{ij} = c_{ji}$.

If one were to introduce new variables $\varepsilon_1, \ldots, \varepsilon_n$, connected with variables $e_1, \ldots, e_n$ by formulas

$$e_i = \varepsilon_i \qquad (i = 1, \ldots, r), \tag{3.16}$$

then we will obtain

$$\sum_{i,j=1}^{n} c_{ij} e_i e_j = \sum_{i=1}^{r} c_i \varepsilon_i^2. \tag{3.17}$$

During different presentations of form (3.14) in the form of (3.15) or (3.17), the number of positive and the number of negative coefficients c_i, and consequently also the number of all coefficients differing from zero are constant. Difference σ between the number π of positive and the number ν of negative coefficients c_i is called the <u>signature</u> of form (3.14). Rank r and signature σ determine simply numbers π and ν, since

$$r = \pi + \nu, \quad \sigma = \pi - \nu.$$

Transformation of quadratic form (3.14) to form (3.15) or (3.17) is called its <u>reduction to sum of squares</u>.

Reduction of quadratic form to sum of squares can be carried out by different methods. As an example, we will expound method offered by Lagrange. Let us consider two cases:

a) during certain k $(1 \leq k \leq n)$ diagonal coefficient c_{kk} is different than zero;

b) all diagonal coefficients c_{ii} are equal to zero.

In the first case quadratic form (3.14) may be represented in the form

$$\sum_{i,j=1}^{n} c_{ij} e_i e_j = \frac{1}{c_{kk}} \left(\sum_{i=1}^{n} c_{ki} e_i \right)^2 + \sum_{i,j=1}^{n} c'_{ij} e_i e_j. \tag{3.18}$$

In the second case — in the form

$$\sum_{i,j=0}^{n} c_{ij} e_i e_j = \frac{1}{2c_{kl}} \left[\sum_{i=1}^{n} (c_{ki} + c_{li}) e_i \right]^2 - \frac{1}{2c_{kl}} \left[\sum_{i=1}^{n} (c_{ki} - c_{li}) e_i \right]^2 + \sum_{i,j=1}^{n} c'_{ij} e_i e_j, \tag{3.19}$$

where c_{kl} is any nondiagonal coefficient differing from zero.

By direct check of equalities (3.18) and (3.19), one can make certain that form

$$\sum_{i,j=1}^{n} c'_{ij} e_i e_j$$

does not contain variable e_k, and form

$$\sum_{i,j=1}^{n} c'_{ij} e_i e_j -$$

variables e_k and e_l.

These forms it is possible to present in the form of analogous sums, etc. As a result of consecutive application of such transformations, form (3.14) is reduced to the sum of the squares.

Real quadratic form (3.14) is called nonnegative (nonpositive), if during any real values of variables e_i ($i = 1, \ldots, n$)

$$\sum_{i,j=1}^{n} c_{ij} e_i e_j \geq 0 \; (\leq 0). \tag{3.20}$$

Criterion of nonnegativeness of quadratic form. So that quadratic form (3.14) is nonnegative, it is necessary and sufficient that all main matrices of its coefficients are nonnegative.

Criterion of nonpositivity of quadratic form. So that quadratic form (3.14) is nonpositive, it is necessary and sufficient that the matrix of its coefficients is such that all its principal minors of even order are nonnegative, and all principal minors of odd order are nonpositive.

Nonnegative (nonpositive) quadratic form is called positively (negatively) determined if sign of equality in inequality (3.20) occurs only during zero values of all variables e_i.

If quadratic form (3.14) is nonnegative (nonpositive), then all coefficients c_i during presentation of it in the form of (3.15) or (3.17) are positive (are negative). Thus, moduli of signatures of nonpositive and nonnegative quadratic forms are equal to their ranks. Nonnegative (nonpositive) quadratic form is positively (negatively) determined if and only if rank r and order n are equal to matrix corresponding to it, i.e., when form is nonsingular.

Criterion of positive determinacy of quadratic form. So that quadratic form (3.14) is positively determined, it is necessary and sufficient that the following inequality is executed

$$D_1 = c_{11} > 0, \; D_2 = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} > 0, \ldots, \; D_n = \begin{vmatrix} c_{11} & \ldots & c_{1n} \\ \cdot & \cdot & \cdot \\ c_{n1} & \ldots & c_{nn} \end{vmatrix} > 0 \tag{3.21}$$

Criterion of negative determinancy of quadratic form. So that quadratic form (3.14) is negatively determined, it is necessary and sufficient that the following

inequality is executed

$$D_1<0,\ D_2>0,\ D_3<0, \dots\ (-1)^n D_n>0. \qquad (3.22)$$

Quadratic form (3.14) always may be thus reduced to a sum of the squares, so that in formulas (3.15) and (3.17) coefficients c_i are determined by equalities

$$c_i=\mu_i \qquad (i=1, \dots, n), \qquad (3.23)$$

where μ_i (i = 1, ..., 0) are characteristic numbers of matrix C.

Such reduction of quadratic form to a sum of squares is called reduction to principal axes.

From the possibility of reducing quadratic form to principal axes, it follows that it is nonnegative (nonpositive) if and only if all characteristic numbers of matrix corresponding to it are nonnegative (nonpositive), and that it is positively (negatively) determined if and only if all characteristic numbers are positive (negative). From the mentioned property of quadratic form, it also follows that its rank is equal to the number of (differing from zero) characteristic numbers of matrix corresponding to it, and the signature is equal to the difference between the number of positive and the number of negative characteristic numbers.

Two real quadratic forms

$$\sum_{i,j=1}^{n} c_{ij}e_ie_j \quad \text{and} \quad \sum_{i,j=1}^{n} d_{ij}e_ie_j \qquad (3.24)$$

determine one parameter family of forms

$$\sum_{i,j=1}^{n} (c_{ij}-\mu d_{ij})\, e_ie_j, \qquad (3.25)$$

where μ is a parameter.

If the second of the shown forms is positively determined, then the parameter family (3.25) is called regular.

Equation

$$\det\| c_{ij}-\mu d_{ij}\|_1^n=0 \qquad (3.26)$$

is called the characteristic equation of the parameter family of forms (3.25), and roots of this equation are called the characteristic numbers of the latter.

Characteristic equation of a regular parameter family of forms always has n real roots.

If parameter family is regular, then by proper change of variables, quadratic forms (3.24) can be simultaneously given to sums of squares of the following form:

$$\left.\begin{array}{l}\sum_{i,j=1}^{n} c_{ij}e_ie_j=\sum_{i=1}^{n}\mu_ie_i^2,\\ \sum_{i,j=1}^{n} d_{ij}e_ie_j=\sum_{i=1}^{n}e_i^2,\end{array}\right| \tag{3.27}$$

where μ_i ($i = 1, \ldots, n$) are mentioned characteristic numbers of a parameter family of forms.

Considering the parameter family of forms (3.25) regular, we will number its characteristic numbers in nondecreasing order:

$$\mu_1 \leqslant \mu_2 \leqslant \ldots \leqslant \mu_n. \tag{3.28}$$

From equality (3.27) follows

$$\frac{\sum_{i,j=1}^{n} c_{ij}e_ie_j}{\sum_{i,j=1}^{n} d_{ij}e_ie_j}=\frac{\mu_1e_1^2+\ldots+\mu_ne_n^2}{e_1^2+\ldots+e_n^2}. \tag{3.29}$$

Equality (3.29) leads to inequalities

$$\mu_1 \leqslant \frac{\sum_{i,j=1}^{n} c_{ij}e_ie_j}{\sum_{i,j=1}^{n} d_{ij}e_ie_j} \leqslant \mu_n, \tag{3.30}$$

where there exist such values of variables e_i ($i = 1, \ldots, n$) when shown equalities are attained. Consequently, <u>the minimum of the considered ratio of forms is equal to characteristic number μ_1, and the maximum to characteristic number</u> μ_n.

If parameter families (3.25) and

$$\sum_{i,j=1}^{n}(\tilde{c}_{ij}-\mu\tilde{d}_{ij})e_ie_j \tag{3.31}$$

are two regular parameter familiar of forms and at any values of magnitudes e_i ($i = 1, \ldots, n$)

$$\frac{\sum_{i,j=1}^{n} c_{ij}e_ie_j}{\sum_{i,j=1}^{n} d_{ij}e_ie_j} \leqslant \frac{\sum_{i,j=1}^{n} \tilde{c}_{ij}e_ie_j}{\sum_{i,j=1}^{n} \tilde{d}_{ij}e_ie_j}, \tag{3.32}$$

then, after designating by $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$ and $\tilde{\mu}_1 \leq \tilde{\mu}_2 \leq \ldots \leq \tilde{\mu}_n$ the corresponding characteristic numbers of these parameter families, we will have

$$\mu_i \leqslant \tilde{\mu}_i \quad (i=1, \ldots, n). \tag{3.33}$$

Hermitian form is a uniform polynomial of the second degree with respect to 2n variables $e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n$, having the form

$$\sum_{i,j=1}^{n} h_{ij} e_i \bar{e}_j, \tag{3.34}$$

coefficients of which h_{ij} are connected by relationships

$$h_{ij} = \bar{h}_{ji} \qquad (i,\ j = 1, \ldots, n). \tag{3.35}$$

Matrix H, composed of coefficients h_{ij} of Hermitian form, is called Hermitian matrix. Rank of matrix H is called also rank of corresponding form.

If all coefficients of Hermitian matrices are real, it takes the form of a real symmetric matrix.

Determinant

$$\det H$$

is called discriminant of form. Hermitian form is called singular if its discriminant is equal to zero.

Hermitian form (3.34) can, by an infinite set of methods, be presented in the form

$$\sum_{i,j=1}^{n} h_{ij} e_i \bar{e}_j = \sum_{i=1}^{r} h_i E_i \bar{E}_i, \tag{3.36}$$

where r is rank of form; $h_i \neq 0$ $(i = 1, \ldots, r)$ – real numbers; $E_i = \sum_{j=1}^{n} \gamma_{ij} e_j$ $(i = 1, \ldots, n)$ – independent complex, linear combinations of variables $e_1, \ldots, e_n$.

Hence, in particular, it follows that Hermitian form takes only real values.

Right side of equality (3.36) is called sum of independent squares.

During the presentation of Hermitian form (3.34) in the form of a sum of independent squares, the number of positive and the number of negative coefficients h_i does not depend on method of presentation. Difference σ between number π of positive and number ν of negative coefficients h_i is called signature of form (3.34). Rank r and signature σ determine simply numbers π and ν.

Hermitian form (3.34) is called nonnegative (nonpositive) if during any values of variables e_i

$$\sum_{i,j=1}^{n} h_{ij} e_i \bar{e}_j \geqslant 0 \ (\leqslant 0). \tag{3.37}$$

Criterion of nonnegativeness of Hermitian form. So that Hermitian form (3.34) is nonnegative, it is necessary and sufficient that all principal minors of matrix

of coefficients are nonnegative.

Criterion of nonpositivity of Hermitian form. So that Hermitian form (3.34) is nonpositive, it is necessary and sufficient that matrix of its coefficients is such that all its principal minors of even order are nonnegative, and all principal minors of odd order are nonpositive.

Nonnegative (nonpositive) Hermitian form is called positively (negatively) determined if sign of equality in inequality (3.37) takes place only during zero values of all variables e_i.

If Hermitian form (3.34) is nonnegative (nonpositive), then all coefficients of it during presentation in the form (3.36) are positive (are negative). Signatures of nonpositive and nonnegative Hermitian forms are equal to their ranks. Nonnegative (nonpositive) Hermitian form is positively (negatively) determined if and only if it is not singular.

Criterion of positive determinancy of Hermitian form. So that Hermitian form (3.34) is positively determined, it is necessary and sufficient that the following inequality is executed:

$$D_1 = h_{11} > 0;\ D_2 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} > 0, \ldots\ D_n = \begin{vmatrix} h_{11} \ldots h_{1n} \\ \ldots\ldots \\ h_{n1} \ldots h_{nn} \end{vmatrix} > 0. \tag{3.38}$$

Criterion of negative determiracy of Hermitian form. So that Hermitian form (3.34) is negatively determined, it is necessary and sufficient that the following inequality is executed:

$$D_1 < 0;\ D_2 > 0;\ D_3 < 0, \ldots\ (-1)^n D_n > 0. \tag{3.39}$$

Hermitian form (3.34) always may be thus converted into a sum of independent squares, so that in formula (3.36) coefficients h_i are determined by equalities

$$h_i = \mu_i \qquad (i = 1, \ldots, n). \tag{3.40}$$

where μ_i ($i = 1, \ldots, n$) are characteristic numbers of matrix H.

Such transformation of Hermitian form is called reduction to principal axes.

Connection of properties of Hermitian form with properties of characteristic numbers of its corresponding matrix is the same as the above-indicated connection of quadratic form with characteristic numbers of its corresponding matrix.

Two Hermitian forms

$$\sum_{i,j=1}^{n} h'_{ij} e_i \bar{e}_j \text{ and } \sum_{i,j=1}^{n} h''_{ij} e_i \bar{e}_j$$

determine parameter family of forms

$$\sum_{i,j=1}^{n} (k'_{ij} - \mu k''_{ij}) e_i \bar{e}_j, \tag{3.41}$$

where μ is a parameter.

If form

$$\sum_{i,j=1}^{n} k'_{ij} e_i \bar{e}_j -$$

is positively determined, then the parameter family is called regular.

Equation

$$\det \| k'_{ij} - \mu k'_{ij} \|_1^n = 0 \tag{3.42}$$

is called characteristic equation of the parameter family of forms (3.41), and the roots of this equation are called the characteristic numbers of the latter.

Characteristic equation of a regular parameter family of forms always has n real roots.

All properties of the relationship of quadratic forms (forming the parameter family) connected with expressions (3.28)-(3.33), stay in force also for the relationship of corresponding Hermitian forms.

Example. In the example considered in the preceding paragraph, two members of differential equations relative to phase coefficients contain as cofactors to Hermitian form

$$2e_1\bar{e}_1 - e_1\bar{e}_2 - \bar{e}_1 e_2 + 2e_2\bar{e}_2.$$

This form corresponds is Hermitian matrix

$$H = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

In this case, all coefficients of matrices are real and, consequently, the matrix is real, symmetric.

Discriminant of form det H is equal to 3. Consequently, rank of matrix is equal to its order, i.e., 2, and form is nonsingular.

Since form factors satisfy conditions (3.38), it is positively determined.

Characteristic numbers of matrix H are roots of equation

$$\begin{vmatrix} 2-\mu & -1 \\ -1 & 2-\mu \end{vmatrix} = 0.$$

whence follows

$$\mu_1 = 1, \ \mu_2 = 3.$$

Reduction of form to principal axes converts it to the form

$$E_1\bar{E}_1 + 3E_2\bar{E}_2.$$

Since both characteristic numbers μ_1 and μ_2 are positive, signature of form is equal to 2.

Introducing into consideration positively determined Hermitian form

$$e_1\bar{e}_1+e_2\bar{e}_2,$$

we will constitute a parameter family of forms

$$(2-\mu)e_1\bar{e}_1-\bar{e}_1e_2-e_1\bar{e}_2+(2-\mu)e_2\bar{e}_2.$$

In construction, this parameter family is regular.

Characteristic equation of parameter family of forms coincides with the above-mentioned equation for determination of characteristic numbers of matrix H. Therefore, characteristic numbers of considered parameter family coincide with characteristic numbers of matrix H.

Relation of investigated Hermitian form to form $e_1\bar{e}_1 + e_2\bar{e}_2$ satisfies inequality

$$1=\mu_1<\frac{2e_1\bar{e}_1-e_1\bar{e}_2-\bar{e}_1e_2+2e_2\bar{e}_2}{e_1\bar{e}_1+e_2\bar{e}_2}<\mu_2=3.$$

§ 7. Boundaries of the Region of Possible Values of Norm of Solutions for Class of Solutions, Determined by Given Initial Value of Norm

In system (3.1) consists of one linear, first order differential equation, functional dependence of norm of its solution on t is simply determined by coefficients of differential equation and initial value of norm r_0. If order of system (3.1) is higher than first, then such uniqueness does not take place; to the same initial value of norm during different initial values of phase coefficients here, in general, correspond different functions r(t).

If one were to modify initial conditions, leaving constant initial norm of solution, then all possible values of functions r(t) on plane (t, r) will occupy a region between the two envelopes of families of these functions. Will designate the function representing the upper envelope by symbol sup r(t) and the function representing the lower envelope by symbol inf r(t).

Exact determination of functions sup r(t) and inf r(t) based on initial value r_0 and coefficients of equation (3.1) is very complex and up to now has still not been solved. However, using relationship (3.6), it is possible, very simply, to establish certain functions S(t) and I(t), estimating functions sup r(t) and inf r(t) from above and from beneath (correspondingly) and coinciding with them in definite particular cases, i.e., functions presenting curves, limiting on plane (t, r) a certain region, either coinciding with region of possible values of functions r(t) or being its expansion.

For determination of functions S(t) and I(t), we will find maxima and minima of right side of equation (3.9) during condition (3.6), which satisfy phase coefficients. The latter problem is solved simply, since right side of equation (3.9) is Hermitian form of phase coefficients. Maximum and minimum values, taken by such a form during condition (3.6), coincide with maximum and minimum values of ratio of forms

$$\frac{\frac{1}{2}\sum_{i=1}^{n}\left(\bar{e}_i\sum_{j=1}^{n}a_{ij}e_j+e_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{e}_j\right)}{\sum_{i=1}^{n}e_i\bar{e}_i}$$

and, consequently (see § 6), with the largest and smallest characteristic numbers of the parameter family of forms

$$\frac{1}{2}\sum_{i=1}^{n}\left(\bar{e}_i\sum_{j=1}^{n}a_{ij}e_j+e_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{e}_j\right)-\mu\sum_{i=1}^{n}e_i\bar{e}_i.$$

The last ones, in turn, coincide with the largest and smallest characteristic numbers of the matrix of form factors

$$\frac{1}{2}\sum_{i=1}^{n}\left(\bar{e}_i\sum_{j=1}^{n}a_{ij}e_j+e_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{e}_j\right),$$

i.e., with the largest and smallest characteristic numbers of the matrix

$$\left\|\frac{a_{ij}+\bar{a}_{ji}}{2}\right\|_1^n.$$

These numbers are roots of equation

$$\det\left\|\frac{a_{ij}+\bar{a}_{ji}}{2}-\delta_{ij}\mu\right\|_1^n=0, \tag{3.43}$$

where δ_{ij} is Kronecker delta ($\delta_{ij}=1$ at $i=j$, $\delta_{ij}=0$ at $i\neq j$).

Let us assume that μ_1 is minimum and μ_n maximum roots of equation (3.43). Then in accordance with above-stated

$$\min_{\sum_{i=1}^{n}e_i\bar{e}_i=1}\frac{\dot{r}}{r}=\mu_1,\quad \max_{\sum_{i=1}^{n}e_i\bar{e}_i=1}\frac{\dot{r}}{r}=\mu_n. \tag{3.44}$$

Consequently, at any t

$$\mu_1\leqslant\frac{\dot{r}}{r}\leqslant\mu_n. \tag{3.45}$$

Hence

$$r_0\exp\int_{t_0}^{t}\mu_1\,dt\leqslant r\leqslant r_0\exp\int_{t_0}^{t}\mu_n\,dt. \tag{3.46}$$

From inequalities (3.46) follows

$$\left.\begin{aligned} I(t) &= r_0 \exp\int_{t_0}^{t} \mu_1 dt, \\ S(t) &= r_0 \exp\int_{t_0}^{t} \mu_n dt. \end{aligned}\right\} \tag{3.47}$$

Example. Canonical expansion of first form of solution of equation

$$\ddot{x} - (t^2 + a)x = 0, \tag{3.48}$$

where a > 0, will convert this equation into system

$$\left.\begin{aligned} \dot{x}_1 &= -\left(\sqrt{t^2+a} + \frac{t}{2(t^2+a)}\right)x_1 + \frac{t}{2(t^2+a)}x_2, \\ \dot{x}_2 &= \frac{t}{2(t^2+a)}x_1 + \left(\sqrt{t^2+a} - \frac{t}{2(t^2+a)}\right)x_2. \end{aligned}\right\} \tag{3.49}$$

Coefficients a_{ij} (i, j = 1, 2) of these equations are real. Therefore, equation (3.43) takes the form

$$\det\|a_{ij} - \delta_{ij}\mu\|_1^n = 0. \tag{3.50}$$

In the considered case will obtain

$$\mu^2 - \frac{t}{t^2+a}\mu - t^2 - a = 0.$$

Solving this equation, we will find

$$\mu_1 = -\frac{t}{2(t^2+a)} - \sqrt{\frac{t^2}{4(t^2+a)^2} + t^2 + a},$$
$$\mu_2 = -\frac{t}{2(t^2+a)} + \sqrt{\frac{t^2}{4(t^2+a)^2} + t^2 + a}.$$

Because of equalities (3.47), taking $t_0 = 0$, we will obtain

$$I(t) = \frac{r_0\sqrt[4]{a}}{\sqrt[4]{a+t^2}} \exp\left[-\int_0^t \sqrt{\frac{t^2}{4(t^2+a)^2} + t^2 + a}\, dt\right],$$
$$S(t) = \frac{r_0\sqrt[4]{a}}{\sqrt[4]{a+t^2}} \exp\int_0^t \sqrt{\frac{t^2}{4(t^2+a)^2} + t^2 + a}\, dt.$$

Graphs of functions I(t) and S(t) at $0 \leq t \leq 1$, a = 1 and $r_0 = 1$ are presented on Fig. 6.

At a = 1 there is known general solution of equation (3.48) [25]:

$$x = \left[C_1 + C_2\int \exp(-t^2)\, dt\right] \exp\frac{t^2}{2}. \tag{3.51}$$

Using given formula and after expanding solution x(t) into canonical components $y_1(t)$ and $y_2(t)$, it is easy to find formula for norm of solution of system (3.49) r(t). Connecting arbitrary constants C_1 and C_2 by condition $r_0 = 1$ according to

this formula, it is possible to construct functions inf r(t) and sup r(t). Graphs of these functions also are shown in Fig. 6. By divergence of functions I(t) and inf r(t), S(t) and sup r(t), it is possible to evaluate accuracy of estimates of I(t) and S(t).

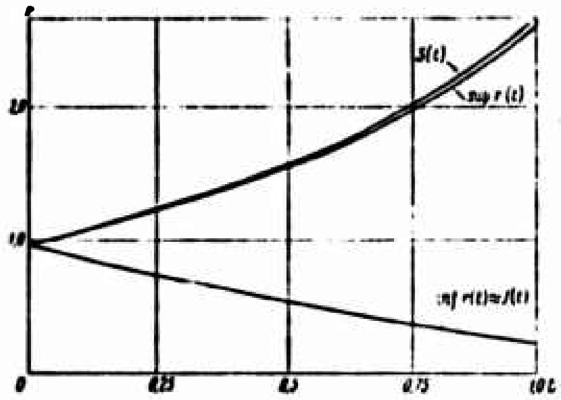

Fig. 6. Boundaries of region of possible values of norm of solution of system (3.49).

§ 8. Systems of Class K

The class (defined in § 1) of considered systems of linear, uniform differential equations, it is possible to separate a subclass, presenting a special interest for the theory of free oscillations expounded in the following chapters. This subclass of systems we will call class K and will define it in the following manner.

Definition: Class K of systems of linear, uniform differential equations of the considered form will be a class of those systems whose coefficients possess the following property: there is possible such transposition of indices for variables x_1, x_2, ..., x_n and, corresponding to it, transposition of indices for coefficients a_{ij} (i, j = 1, ..., n) at which it is possible to unite all indices into two groups, including in the first indices 1, 2, ..., n - 2m, and in the second — the others, in such a manner that there are executed relationships $a_{ij} = \overline{a}_{kl}$ ($i, j \leq n - 2m$) where k = i if i is index of first group; k = i + m if i is index of second group; l = j + m, if j is index of first group; l = j + m, if j is index of second group.

Sum of diagonal elements of matrix A, as is known, is called its trace and is designated by symbol Sp A:

$$\mathrm{Sp}\,A = \sum_{i=1}^{n} a_{ii}. \tag{3.50}$$

From the given determination, it follows that in the case of a system of class K trace of matrix A is real. Consequently, the full system of coefficients a_{ii} of matrix A

$$a_{11}, a_{22}, \ldots, a_{nn}$$

in this case consists of real elements and complex conjugate pairs.

It is possible to introduce many examples of systems belonging to class K. We will describe, in particular, systems with real coefficients and systems whose matrices of coefficients are Hermitian. For instance, the system (3.49) considered in the example in § 7 is a system of class K.

§ 9. Functions $f_1(t), \ldots, f_n(t)$ and System of Equations Which They Satisfy

Solutions of a system of equations relative to phase coefficients, in general, are complex even if initial values of phase coefficients are real. If initial system of equations belongs to class K, then with the help of a simple change of variables from its corresponding system of equations relative to phase coefficients, it is possible to cross to a new system, all solutions of which, corresponding to real initial values of variables, are real.

The new system we will find, introducing variables $f_1, \ldots, f_n$, after connecting them with variables $e_1, \ldots, e_n$ by the following dependences:

$e_l = f_l$ — for indices l, corresponding to real coefficients a_{ll};

$$\left.\begin{aligned} e_j &= \frac{1}{\sqrt{2}}(f_j + if_k) \\ e_k &= \frac{1}{\sqrt{2}}(f_j - if_k) \\ (i &= \sqrt{-1}) \end{aligned}\right\} \text{— for indices } j \text{ and } k, \text{ corresponding to complex conjugate coefficients } a_{jj} \text{ and } a_{kk}. \quad (3.53)$$

It is not difficult to see that variables $f_1, \ldots, f_n$ satisfy condition

$$\sum_{j=1}^{n} f_j^2 = 1, \quad (3.54)$$

if variables $e_1, \ldots, e_n$ are connected by condition (3.6).

After carrying out substitution (3.53) in system (3.12) we will obtain, after simple transformations, a system of differential equations relative to variables $f_1, \ldots, f_n$ in the form

$$\dot{f}_i = \sum_{j=1}^{n} p_{ij} f_j \quad (i = 1, \ldots, n). \quad (3.55)$$

After applying indices 1, 2, ..., n - 2m to real roots, and indices n - 2m + 1 and n - 2m + 2, ..., n - 1 and n to complex conjugate pairs, we will obtain formula for coefficients p_{ij} in the form

$$\left.\begin{array}{ll}
p_{ij} = a_{ij} - \delta_{ij}F & \text{at } i, j \leqslant n-2m; \\
\left.\begin{array}{l} p_{ij} = \sqrt{2}\,\mathrm{Re}\, a_{ij} \\ p_{i,j+1} = -\sqrt{2}\,\mathrm{Im}\, a_{ij} \end{array}\right\} & \begin{array}{l}\text{at } i \leqslant n-2m, \\ j = n-2m+1,\ n-2m+3, \ldots, n-1; \end{array} \\
\left.\begin{array}{l} p_{ij} = \frac{1}{\sqrt{2}}\,\mathrm{Re}\, a_{ij}, \\ p_{i+1,j} = \frac{1}{\sqrt{2}}\,\mathrm{Im}\, a_{ij} \end{array}\right\} & \begin{array}{l}\text{at } i = n-2m+1,\ n-2m+3, \ldots \\ \ldots, n-1,\ j \leqslant n-2m; \end{array} \\
\left.\begin{array}{l} p_{ij} = \mathrm{Re}\,(a_{ij} + a_{i,j+1}) - \delta_{ij}F, \\ p_{i,j+1} = -\mathrm{Im}\,(a_{ij} - a_{i,j+1}), \\ p_{i+1,j} = \mathrm{Im}\,(a_{ij} + a_{i,j+1}), \\ p_{i+1,j+1} = \mathrm{Re}\,(a_{ij} - a_{i,j+1}) - \delta_{ij}F. \end{array}\right\} & \begin{array}{l}\text{at } i, j = n-2m+1, \\ n-2m+3, \ldots, n-1; \end{array}
\end{array}\right\} \qquad (3.56)$$

where δ_{ij} is Kronecker delta; F — quadratic form relative to variables $f_1, \ldots, f_n$ obtained from Hermitian form

$$\frac{1}{2}\sum_{i=1}^{n}\left(e_i\sum_{j=1}^{n}a_{ij}e_j + e_i\sum_{j=1}^{n}\bar{a}_{ij}\bar{e}_j\right)$$

with the help of substitutions (3.53).

From formulas (3.56) it follows that all coefficients of system (3.55) are real. Therefore, if initial values of variables $f_1, \ldots, f_n$ are real, then their subsequent values also are real.

Consequently, system (3.55) has real solutions. These solutions correspond to solutions of system (3.12), in which variables e_i, corresponding in index to real coefficients a_{ii}, are real, and variables e_j and e_k, corresponding in indices to complex conjugate coefficients a_{jj} and a_{kk}, are complex conjugate.

Example. Canonical components y_1 and y_2 of real solution of equation

$$\ddot{x} + ix = 0 \qquad (3.57)$$

are complex. Corresponding to them, phase coefficients also are complex. Roots of algebraic equation

$$\lambda^2 + i = 0$$

are

$$\lambda_{1,2} = \pm i\sqrt{i},$$

and coefficients g_{jk} (j, k = 1, 2) are expressed by formulas [see equation (2.34)]

$$g_{11} = -\frac{1}{4i},\ g_{12} = \frac{1}{4i},\ g_{21} = \frac{1}{4i},\ g_{22} = -\frac{1}{4i}.$$

After changing designations y_1 and y_2 to x_1 and x_2, we will write system of equations relative to canonical components in the form

$$\left.\begin{aligned}\dot{x}_1 = a_{11}x_1 + a_{12}x_2,\\ \dot{x}_2 = a_{21}x_1 + a_{22}x_2,\end{aligned}\right\}$$

where

$$a_{11} = \lambda_1 + \varepsilon_{11} = i\sqrt{\bar{\imath}} - \frac{1}{4t};$$

$$a_{12} = \varepsilon_{12} = \frac{1}{4t};$$

$$a_{21} = \varepsilon_{21} = \frac{1}{4t};$$

$$a_{22} = \lambda_2 + \varepsilon_{22} = -i\sqrt{\bar{\imath}} - \frac{1}{4t}.$$

Form F is expressed through variables e_1 and e_2 in the form

$$F = \frac{1}{4t}(e_1^2 - 2e_1e_2 + e_2^2) = \frac{(e_1 - e_2)^2}{4t}.$$

Changing to variables f_1 and f_2 according to formulas (3.53), we obtain

$$F = -\frac{f_2^2}{2t}.$$

By formulas (3.56) we will find coefficients p_{ij} $(i, j = 1, 2)$,

$$p_{11} = \frac{f_2^2}{2t};\quad p_{12} = -\sqrt{\bar{\imath}};\quad p_{21} = +\sqrt{\bar{\imath}};\quad p_{22} = -\frac{1}{2t} + \frac{f_2^2}{2t} = -\frac{f_1^2}{2t}.$$

Equations (3.55) will take the form

$$\left.\begin{aligned}\dot{f}_1 = \frac{f_1f_2^2}{2t} - \sqrt{\bar{\imath}}f_2,\\ \dot{f}_2 = \sqrt{\bar{\imath}}f_1 - \frac{f_1^2f_2}{2t}.\end{aligned}\right| \qquad (3.58)$$

§ 10. Majorants and Minorants of Functions $f_1(t), \ldots, f_n(t)$

In this and subsequent sections we will consider real particular solutions of a system of differential equations (3.55), assuming, consequently, that initial system of differential equations belongs to class K. Will consider only such solutions for which there is executed condition (3.54).

In § 7 there were established estimates for boundaries of the region of possible values of function r(t) on the assumption that initial values of variables $x_1, \ldots, x_n$ can take any values in region, determined by condition

$$r(t_0) = r_0. \qquad (3.59)$$

Obviously, if one were to set up any definite initial values from the region given by condition (3.59), then function r(t) will take during every value of t only one value, i.e., region of possible values of function r(t) on plane (t, r) will

degenerate into a line. There appears problem how to establish more exact appraisals for function $r(t)$, using information about initial conditions.

This problem is subsequently expanded. We will look for an appraisal not only for function $r(t)$ but also for functions $f_j(t)$ $(j = 1, \ldots, n)$ and function $x_j(t)$. It turns out to be convenient to start from functions $f_j(t)$. Inasmuch as these functions are assumed to be real, appraisal functions for them it is possible to seek in the form of a majorant and a minorant.

Function Min $f_j(t)$ we will call minorant and function Maj $f_j(t)$ majorant of function $f_j(t)$ if for all t $(t_0 < t < \infty)$

$$\mathrm{Min}\, f_j(t) \leqslant f_j(t),$$
$$\mathrm{Maj}\, f_j(t) \geqslant f_j(t).$$

In this section we will construct sequences of minorants $\mathrm{Min}_1 f_j(t)$, $\mathrm{Min}_2 f_j(t) \ldots$ and majorants $\mathrm{Maj}_1 f_j(t)$, $\mathrm{Maj}_2 f_j(t)$, ... of functions $f_j(t)$, connected among themselves and with functions $f_j(t)$ by relationships

$$\mathrm{Min}_1 f_j(t) \leqslant \mathrm{Min}_2 f_j(t) \leqslant \ldots \leqslant f_j(t) \leqslant \ldots \leqslant \mathrm{Maj}_2 f_j(t) \leqslant \leqslant \mathrm{Maj}_1 f_j(t) \tag{3.60}$$

and approaching in some finite interval during growth of their ordinal number to functions $f_j(t)$.

We will construct these sequences in the following form.

Let us turn to equations (3.55). Coefficients p_{jk} of these equations during $j \neq k$ are known magnitudes. Coefficients p_{jj} depend on quadratic form $F(f_1, \ldots, f_n)$. Since magnitude F itself is a function of solution of system (3.55), then coefficients p_{jj} are unknown magnitudes.

Essential for the construction of estimates of the solution of the system (3.55) are the property (3.54) of variables $f_1, \ldots, f_n$ and, connected with it, the limitedness of function $F(f_1, \ldots, f_n)$ during arbitrary values of arguments for each value of t. On the basis of these properties, it is possible to indicate final upper bounds for moduli of right sides of equations (3.55). Using inequality of Cauchy [34], we will find

$$\left|\sum_{k=1}^{n} p_{jk} f_k\right| \leqslant \sqrt{p_{j1}^2 + \cdots + p_{jn}^2} \quad (j = 1, \ldots, n).$$

Taking into account equations (3.55), we will obtain

$$-\sqrt{p_{j1}^2 + \cdots + p_{jn}^2} \leqslant \dot{f}_j(t) \leqslant \sqrt{p_{j1}^2 + \cdots + p_{jn}^2} \quad (j = 1, \ldots, n).$$

From this inequality follows

$$-\int_{t_0}^{t}\sqrt{p_{j1}^2+\dots+p_{jn}^2}\,dt \leqslant f_j(t)-f_j(t_0) \leqslant \int_{t_0}^{t}\sqrt{p_{j1}^2+\dots+p_{jn}^2}\,dt \qquad (j=1,\dots,n). \tag{3.61}$$

Since coefficients p_{jj} depend on quadratic form $F(f_1, \dots, f_n)$, then left and right sides of the last inequalities also are its functions. Taking into account equivalence of conditions (3.54) and (3.6), on the basis of that presented in § 7, we will obtain, for appraisal of form F, inequality

$$\mu_1 < F < \mu_2. \tag{3.62}$$

Inequalities (3.61), obviously, will remain in force if, instead of true values of magnitudes p_{jj}^2, we substitute their maximum possible values, found under the condition that form F takes arbitrary values in region (3.62). Such maximum possible values must be values at which either $F = \mu_1$ or $F = \mu_2$.

We will designate by symbols $\sigma_j^{(1)}$ ($j = 1, \dots, n$) magnitudes

$$\sqrt{p_{j1}^2+\dots+p_{jn}^2} \qquad (j=1\dots n),$$

attained during replacement in expressions, under the radical, of true values of component p_{jj}^2 by their above-determined, maximum possible values; we will designate by symbols $\rho_j^{(1)}$ magnitudes inverse to them in sign. Considering the introduced designations, we will obtain

$$f_j(t_0)+\int_{t_0}^{t}\rho_j^{(1)}(t)\,dt < f_j(t) < f_j(t_0)+\int_{t_0}^{t}\sigma_j^{(1)}(t)\,dt. \tag{3.63}$$

Because of inequalities (3.63) it is possible to assume

$$\left.\begin{aligned} \mathrm{Min}_1 f_j(t) &= f_j(t_0)+\int_{t_0}^{t}\rho_j^{(1)}(t)\,dt, \\ \mathrm{Maj}_1 f_j(t) &= f_j(t_0)+\int_{t_0}^{t}\sigma_j^{(1)}(t)\,dt \end{aligned}\right\} \qquad (j=1,\dots,n). \tag{3.64}$$

By equalities (3.64) we will determine the first members of sequences constructed by us. For determination of remaining members of sequences, we will introduce the following recurrence dependences:

$$\left.\begin{array}{l}\mu_1^{(s)}=\min F,\\ \mu_n^{(s)}=\max F\end{array}\right\}\text{— during conditions (3.54) and } \operatorname{Min}_{s-1} f_j(t) \leq f_j(t) \leq \operatorname{Maj}_{s-1} f_j(t) \quad (j=1,\dots,n);$$

$$\left.\begin{array}{l}\rho_j^{(s)}=\min \sum_{k=1}^{n} p_{jk} f_k,\\ \sigma_j^{(s)}=\max \sum_{k=1}^{n} p_{jk} f_k\\ (j=1,\dots,n)\end{array}\right|\begin{array}{l}\text{— during conditions}\\ \text{a) for form } F \;\; \mu_1^{(s)} < F < \mu_n^{(s)},\\ \text{b) for functions } f_k(t)\\ \operatorname{Min}_{s-1} f_k(t) \leq f_k(t) < \operatorname{Maj}_{s-1} f_k(t)\\ (k=1,\dots,n) \text{ и } (3.54);\end{array} \qquad (3.65)$$

$$\operatorname{Min}_s f_j(t)=f_j(t_0)+\int_{t_0}^{t} \rho_j^{(s)}(t)\,dt$$

$$\operatorname{Maj}_s f_j(t)=f_j(t_0)+\int_{t_0}^{t} \sigma_j^{(s)}(t)\,dt$$

$$(s=2,3\dots)$$

Sequences, built on diagram (3.65), satisfy condition (3.60), since range of values of variables $f_1, \dots, f_n$ for which are determined maximum and minimum values of magnitudes F and $\sum_{k=1}^{n} p_{jk} f_k \; (j=1,\dots,n)$, according to conditions of construction cannot be expanded (either they are narrowed or remain constant). We will show that they possess also another required property: they approach in any finite interval (t_0, T) to functions $f_j(t)$.

We will define function $\nu_s(t)$ $(s = 1, 2, \dots)$ by equality

$$\nu_s(t)=\max_j\left[\operatorname{Maj}_s f_j(t)-\operatorname{Min}_s f_j(t)\right]. \qquad (3.66)$$

Obviously, all sequences $\{\operatorname{Maj}_s f_j(t)\}$ and $\{\operatorname{Min}_s f_s(t)\}$ approach in a finite interval to functions $f_j(t)$ $(j = 1, \dots, n)$ if and only if, in this interval, functions $\nu_s(t)$ during $s \to \infty$ approach to zero. We will prove that functions $\nu_s(t)$ possess these properties.

Let us assume that $p(t)$ is the biggest of magnitudes $|p_{jk} + \delta_{jk}F|$ $(j, k = 1, \dots, n)$. Then for difference of magnitudes $\sigma_j^{(s+1)}$ and $\rho_j^{(s+1)}$, determined by equations (3.65), we will obtain appraisal

$$\sigma_j^{(s+1)}-\rho_j^{(s+1)} < np(t)\,\nu_s(t)+(Ff_j)_{max}-(Ff_j)_{min}, \qquad (3.67)$$

where $(Ff_j)_{max}$ and $(Ff_j)_{min}$ are maximum and minimum values of product Ff_j during conditions

a) for first cofactor

$$\mu_1^{(s+1)} < F < \mu_n^{(s+1)},$$

b) for second cofactor

$$\operatorname{Min}_s f_j(t) < f_j(t) \leq \operatorname{Maj}_s f_j(t)$$

and condition (3.54).

Let us assume that F'_j and f'_j ($j = 1, \ldots, n$) are values of magnitudes F and f_j for which

$$F_j f'_j = (F f_j)_{\min} \qquad (j = 1, \ldots, n),$$

and F''_j and f''_j ($j = 1, \ldots, n$) are values of the same magnitudes, for which

$$F''_j f''_j = (F f_j)_{\max} \qquad (j = 1, \ldots, n).$$

Then

$$(F f_j)_{\max} - (F f_j)_{\min} = F''_j f''_j - F'_j f'_j = F''_j (f''_j - f'_j) \pm \Delta F_j f'_j,$$

where

$$\Delta F_j = |F''_j - F'_j|.$$

After designating by symbol M(t) the maximum of modulus of form F during condition (3.54) and by symbol b(t) the biggest of magnitudes $|\frac{\partial F}{\partial f_j}|$ ($j = 1, \ldots, n$) during condition (3.54), on the basis of found equality we will obtain

$$(F f_j)_{\max} - (F f_j)_{\min} \leqslant [M(t) + nb(t)] \nu_s(t).$$

Hence because of inequality (3.67)

$$\rho_j^{(s+1)} - \rho_j^{(s+1)} \leqslant (n[p(t) + b(t)] + M(t)) \nu_s(t) \qquad (j = 1, \ldots, n)$$

and, taking into account determinations (3.65) and (3.66),

$$\nu_{s+1}(t) < \int_{t_0}^{t} (n[p(t) + b(t)] + M(t)) \nu_s(t)\, dt.$$

Let us assume that

$$H = \max (n[p(t) + b(t)] + M(t)) \qquad (t_0 \leqslant t \leqslant T).$$

Then

$$\nu_{s+1}(t) < \int_{t_0}^{t} H\nu_s(t)\, dt \leqslant (t - t_0) H \nu_s(t) \qquad (t_0 \leqslant t < T), \tag{3.68}$$

since function $\nu_s(t)$ is nondecreasing.

From inequality (3.68) it follows that on section $[t_0, t_0 + \tau]$, where τ is any number, satisfying condition $0 < \tau < \frac{1}{H}$,

$$\nu_{s+1}(t) < \nu_s(t).$$

Therefore sequence $\nu_1(t)$, $\nu_2(t)$, ... on section $[t_0, t_0 + \tau]$ monotonically approaches a certain limit. This limit is zero since, because of inequality

$$\frac{\nu_{s+1}}{\nu_s} < H(t - t_0) \qquad (t_0 < t < T)$$

during any values of t from section $[t_0, t_0 + \tau]$ each member of sequence $\frac{\nu_1(t)}{\nu_1(t)}$, $\frac{\nu_2(t)}{\nu_1(t)}$, ... is less than corresponding member of sequence 1, $H\tau$, $H^2\tau^2$, ..., limit of

which is zero.

In order to prove that sequence $\{\nu_s(t)\}$ approaches zero in whole interval (t_0, T), we will present half-open interval $(t_0, T]$ in the form of the sum of half-open intervals

$$(t_0, t^{(1)}], \ (t^{(1)}, t^{(2)}], \dots, (t^{(q-1)}, t^{(q)}], \dots, (t^{(N-1)}, T],$$

where

$$t^{(q)} = t_0 + \frac{qT}{N};$$

N is arbitrary natural number, satisfying inequality

$$N > TH.$$

Let us assume that considered property in q-th interval takes place. Then for any positive ε there exists such $s_0 = s_0(\varepsilon)$, that for all $s \geq s_0$

$$\nu_s(t^{(q)}) < \varepsilon.$$

For magnitudes $\nu_{s+1}(t)$ in q + 1st interval we will obtain

$$\nu_{s+1}(t) < \varepsilon + H \int_{t^{(q)}}^{t} \nu_s(t)\,dt < \varepsilon + (t - t^{(q)}) H \nu_s(t)$$
$$(t^{(q)} < t \leq t^{(q+1)}). \tag{3.69}$$

For arbitrary value of t from q + 1st interval, on the basis of inequality (3.69), there can be obtained

$$\nu_{s+k+1} < \varepsilon + C\nu_{s+k} < \varepsilon(1 + C + \dots + C^{k}) + C^{k}\nu_s < \frac{\varepsilon}{1-C} + C^{k}\nu_s, \tag{3.70}$$

where

$$C = \frac{T}{N} H < 1.$$

From inequality (3.70) it follows that during sufficiently small ε, sequence $\{\nu_s\}$ is diminishing. Its limit is not larger than magnitude $\frac{\varepsilon}{1 - C}$. Since magnitude ε is possible to order arbitrarily, this limit is equal to zero.

From that proven it follows that $\nu_s(t) \to 0$ during $s \to \infty (t^{(q)} \leq t \leq t^{(q+1)})$ during any q. And this means that sequences $\{\nu_s(t)\}$ approach zero in whole interval (t_0, T). This also was necessary to prove.

Example. For system of equations (3.49) (see § 7) functions $\mu_1(t)$ and $\mu_n(t)$ have the following form:

$$\mu_1 = -\frac{t}{2(t^2+a)} - \sqrt{\frac{t^2}{4(t^2+a)^2} + t^2 + a};$$
$$\mu_2 = -\frac{t}{2(t^2+a)} + \sqrt{\frac{t^2}{4(t^2+a)^2} + t^2 - a}.$$

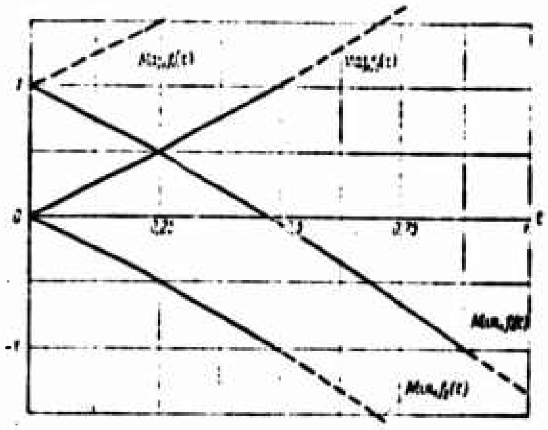

Fig. 7. Majorants and minorants of functions $f_1(t)$ and $f_2(t)$.

Considering a = 1, we will consider particular solution of this system with initial conditions

$$e_1 = 1,\ e_2 = 0,\ \tau = 1.$$

Because of the realness of the coefficients of system (3.49) we have

$$e_1(t) = f_1(t),\ e_2(t) = f_2(t).$$

In view of the assumed, initial conditions functions $f_1(t)$ and $f_2(t)$ are real.

We will define $\mathrm{Min}_1 f_1(t)$, $\mathrm{Maj}_1 f_1(t)$, $\mathrm{Min}_1 f_2(t)$ and $\mathrm{Maj}_1 f_2(t)$.

Turning to system of equations relative to phase coefficients, built for considered initial equation, and presenting it in the form (3.55), we will obtain for coefficients p_{jk} (j , k = 1, 2), according to the formula (3.56),

$$p_{11} = -\sqrt{t^2+1} - \frac{t}{2\sqrt{t^2+1}} - F,$$

$$p_{12} = \frac{t}{2\sqrt{t^2+1}},$$

$$p_{21} = \frac{t}{2\sqrt{t^2+1}},$$

$$p_{22} = \sqrt{t^2+1} - \frac{t}{2\sqrt{t^2+1}} - F.$$

Let us note that coefficient p_{11} numerically is maximum, when $F = \mu_1$ and coefficient p_{22} numerically is maximum, when $F = \mu_2$. Therefore,

$$\rho_1^{(1)} = -\sqrt{\left(\sqrt{t^2+1} + \frac{t}{t^2+1} - \sqrt{\frac{t^2}{4(t^2+1)^2} + t^2+1}\right)^2 + \frac{t^2}{4(t^2+1)^2}} = -\sigma_1^{(1)};$$

$$\rho_2^{(1)} = -\sqrt{\left(\sqrt{t^2+1} - \frac{t}{t^2+1} + \sqrt{\frac{t^2}{4(t^2+1)^2} + t^2+1}\right)^2 + \frac{t^2}{4(t^2+1)^2}} = -\sigma_2^{(1)}.$$

Hence by formulas (3.64) we will find unknown magnitudes.

On Fig. 7 are built graphs of functions $\mathrm{Min}_1 f_1(t)$, $\mathrm{Maj}_1 f_1(t)$, $\mathrm{Min}_1 f_2(t)$ and $\mathrm{Maj}_1 f_2(t)$ on section $0 \le t \le 1$.

§ 11. Majorant and Minorants of Norm of Particular Solution

Using found appraisals for functions f_j ($j = 1, \ldots, n$), equation (3.9), characterizing connection between logarithmic derivative of norm of solution and its phase coefficients, and initial conditions of investigated particular solution, it is easy to construct majorants and minorants of function $r(t)$.

Producing in equation (3.9) replacement of variables $e_1, \ldots, e_n$ by variables $f_1, \ldots, f_n$ in accordance with designations taken in § 9, we will copy this equation in the form

$$\frac{d \ln r}{dt} = F(f_1, \ldots, f_n). \tag{3.71}$$

Taking into consideration condition (3.54) and found appraisals for magnitude F [see formulas (3.62) and (3.65)], from formula (3.71) we immediately will obtain

$$\begin{gathered} \mathrm{Min}_k r(t) = r_0 \exp \int_{t_0}^{t} \mu_1^{(k)}(t)\,dt, \\ \mathrm{Maj}_k r(t) = r_0 \exp \int_{t_0}^{t} \mu_n^{(k)}(t)\,dt \\ (k = 1, 2, \ldots), \end{gathered} \tag{3.72}$$

where

$$\mu_1^{(1)} = \mu_1, \ldots, \mu_n^{(1)} = \mu_n.$$

Having compared formulas (3.72) with formulas (3.47), let us note that

$$\left.\begin{aligned} \mathrm{Min}_1 r(t) &= I(t), \\ \mathrm{Maj}_1 r(t) &= S(t). \end{aligned}\right\} \tag{3.73}$$

Consequently, during given value r_0 first members of sequences of majorants and minorants of function $r(t)$ for particular solution coincide with (fixed in § 7) upper and lower appraisals of function $r(t)$ for class of solutions given by initial value of norm.

Between the following members of sequences, obviously, there take place relationships

$$\left.\begin{gathered} \mathrm{Min}_k r(t) \leqslant \mathrm{Min}_{k+1} r(t), \\ \mathrm{Maj}_k r(t) \geqslant \mathrm{Maj}_{k+1} r(t) \end{gathered}\right\} \\ (k = 1, 2, \ldots).$$

Sequences $\mathrm{Min}_1 r(t)$, $\mathrm{Min}_2 r(t)$, ..., $\mathrm{Maj}_1 r(t)$, $\mathrm{Maj}_2 r(t)$, ... evenly converge to function $r(t)$ in any finite interval (t_0, T).

Example: We will define first and second members of sequences of majorants and minorants of norm of solution $r(t)$ of system (3.49) during $a = 1$, $t_0 = 0$, $r_0 = 1$, $f_1(0) = 1$ $[f_2(0) = 0]$.

Functions $\mathrm{Min}_1 r(t)$ and $\mathrm{Maj}_1 r(t)$, according to above-stated, coincide with functions $I(t)$ and $S(t)$ determined for initial conditions $t_0 = 0, R_0 = L$. Functions $I(t)$ and $S(t)$ during shown initial conditions were determined in Section 7 (see Fig. 6).

Using results of calculation of functions $\mathrm{Min}_1 f_1(t)$, $\mathrm{Maj}_1 f_1(t)$, $\mathrm{Min}_1 f_2(t)$, $\mathrm{Maj}_1 f_2(t)$ (see Fig. 7), carried out for considered initial conditions in preceding section, will define functions $\mathrm{Min}_2 r(t)$ and $\mathrm{Maj}_2 r(t)$.

In Table 1 are given values of coefficients f_1 and f_2 corresponding to extreme values of form F.

Table 1.

Conditions of extremum	t	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
$F=\mu_1$	$\pm f_1$	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00
	$\pm f_2$	0,00	0,02	0,03	0,04	0,06	0,08	0,09	0,10	0,10	0,09	0,09
$F=\mu_2$	$\mp f_1$	0,00	0,02	0,05	0,07	0,08	0,09	0,09	0,09	0,09	0,09	0,09
	$\mp f_2$	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	0,99	0,99	1,00

Having compared table data with graphs shown in Fig. 7, it is easy to conclude that values of coefficients $f_1(t)$ and $f_2(t)$ giving minimum to form $F(F = \mu_1)$, in whole interval (0.1) are included in regions limited by their majorants and minorants. Consequently,

$$\mathrm{Min}_2\, r(t) = I(t).$$

From analogous comparison, it follows that value of coefficients $f_1(t)$ and $f_2(t)$ giving maximum to form $F(F = \mu_2)$, are included between majorants and minorants of these coefficients only during $t \geq 0.5$. During $t \leq 0.4$ both coefficients exceed the bounds of shown regions.

Consequently, during $t \geq 0.5$

$$\mu_2^{(2)} = \mu_2^{(1)} = \mu_2.$$

During $t \leq 0.4$ form F attains maximum, when $f_2 = f_2^{(0)}$, where

$$f_2^{(0)} = \mathrm{Min}_1 f_2(t).$$

Consequently, during $t \leq 0.4$

$$\mu_2^{(2)} = F\left[\sqrt{1-(\mathrm{Min}_1 f_2)^2},\ \mathrm{Min}_1 f_1\right].$$

Graph of function $\mathrm{Maj}_2 r(t)$, built during found values of coefficient $\mu_2^{(2)}$ by the formulas (3.72), is represented on Fig. 8. In the same place are given graphs of functions $\mathrm{Min}_1 r(t)$, $\mathrm{Maj}_1 r(t)$, $\mathrm{Min}_2 r(t)$ and graphs of function $r(t)$, built as is shown in § 7, on the formula given there, of general solution of equation (3.48).

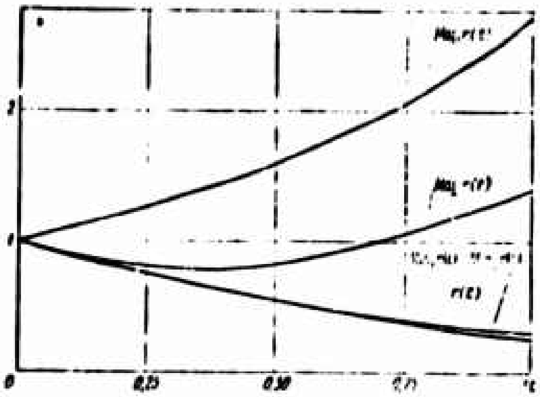

Fig. 8. Majorants and minorants of norm of particular solution of system (3.49).

§ 12. Estimates of Coordinates of Particular Solution

By definition, coordinates of solution $x_1, \ldots, x_n$ are connected with norm and and phase coefficients of solution by dependence (3.4)

$$x_i = r e_i$$
$$(i = 1, \ldots, n).$$

Using formulas (3.4) and (3.53), we will obtain formulas connecting coordinates of solution with norm of solution by functions $f_1, \ldots, f_n$. They have the form

$x_l = r f_l$ — for indices l corresponding to real coefficients a_{ll};

$$\left.\begin{aligned} x_j &= \frac{r}{\sqrt{2}}(f_j + i f_k), \\ x_k &= \frac{r}{\sqrt{2}}(f_j - i f_k) \\ (i &= \sqrt{-1}) \end{aligned}\right\}$$ — for indices j and k corresponding to complex conjugate coefficients a_{jj} and a_{kk}. (3.74)

Assuming that initial values of functions $f_i(t)$ $(i = 1, \ldots, n)$ are real, applying majorant and minorant appraisal, found in preceding sections, for these functions and function $r(t)$, and considering that, because of condition (3.54), region of possible values of functions $f_i(t)$ is limited by section $[-1, 1]$, with the help of formulas (3.54) we will find

a) for indices i corresponding to real coefficients a_{ii}

b) for indices j and k corresponding to complex conjugate coefficients a_{jj} and a_{kk},

$$\left.\begin{aligned}
&\min[\mathrm{Min}_m r \max(\mathrm{Min}_l f_i, -1),\\
&\quad \mathrm{Maj}_m r \max(\mathrm{Min}_l f_i, -1)] < \\
&\quad < x_i \leqslant \max[\mathrm{Min}_m r \min(\mathrm{Maj}_l f_i, 1),\\
&\quad \mathrm{Maj}_m r \min(\mathrm{Maj}_l f_i, 1)];\\
&\mathrm{Maj}_m r \max(\mathrm{Min}_l f_j, -1)] \leqslant \mathrm{Re}\, x_j = \mathrm{Re}\, x_k < \frac{1}{\sqrt{2}} \min[\mathrm{Min}_m r \max(\mathrm{Min}_l f_j, -1),\\
&\quad \leqslant \frac{1}{\sqrt{2}} \max[\mathrm{Min}_m r \min(\mathrm{Maj}_l f_j, 1),\\
&\quad \mathrm{Maj}_m r \min(\mathrm{Maj}_l f_j, 1)],\\
&\quad \frac{1}{\sqrt{2}} \min[\mathrm{Min}_m r \max(\mathrm{Min}_l f_k, -1),\\
&\quad \mathrm{Maj}_m r \max(\mathrm{Min}_l f_k, -1)] \leqslant \mathrm{Im}\, x_j = -\mathrm{Im}\, x_k \leqslant\\
&\leqslant \frac{1}{\sqrt{2}} \max[\mathrm{Min}_m r \min(\mathrm{Maj}_l f_k, 1), \quad \mathrm{Maj}_m r \min(\mathrm{Maj}_l f_k, 1)].
\end{aligned}\right\} \quad (3.75)$$

Considering majorants and minorants of functions $f_i(t)$ ($i = 1, \ldots, n$) of first, second, ..., l-th approximation and majorants and minorants of function r(t) of first, second, ..., m-th approximation, considering $m = l$ or $m = l + 1$ and increasing l, we will obtain sequence of inequalities (3.75) and, consequently, sequence of appraisals of coordinates of solution. Elements of the latter – estimates of coordinates of solution – converge to solution during $l \to \infty$.

Example: In §§ 10 and 11 is shown construction of majorants and minorants of first approximation of functions $f_1(t)$ and $f_2(t)$ and majorants and minorants of first and second approximations of function r(t) in interval (0.1) for particular solution of system (3.49) (see § 7) during a = 1 and following initial conditions:

$$e_1(0) = f_1(0) = 1;\ e_2(0) = f_2(0) = 0;\ r(0) = 1.$$

Using results of these constructions (see Figs. 7-8) and considering $l = 1$, $m = 1$ and $l = 1$, $m = 2$, on the basis of inequalities (3.75) we will obtain two systems of appraisals for coordinates of solution x_1 and x_2. Due to realness of the latter (ensuing from realness of coefficients a_{11} and a_{22}) appraisals will have the character of majorants and minorants. These appraisals are presented by graphs on Fig. 9. There is designated $\mathrm{Min}_1 x_{1,2}$ and $\mathrm{Maj}_{1,2}$ – lower and upper appraisal of coordinate $x_{1,2}$, found during condition $l = 1$, $m = 1$; $\mathrm{Min}_2 x_{1,2}$ and $\mathrm{Maj}_2 x_{1,2}$ – lower and upper appraisal of coordinate $x_{1,2}$, found during condition $l = 1$, $m = 2$.

On Fig. 9 also are depicted graphs of functions $x_1(t)$ and $x_2(t)$, appraisal of which it was necessary to find. These functions are calculated for the considered initial conditions by the method shown in § 7.

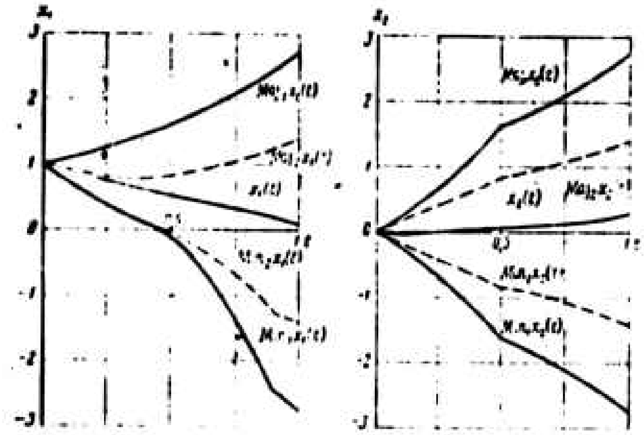

Fig. 9. Majorants and minorants of functions $x_1(t)$ and $x_2(t)$.

CHAPTER IV

ANALYSIS OF FREE OSCILLATIONS IN A FINITE INTERVAL OF TIME

In this chapter is expounded a series of methods of analysis of free oscillations of linear systems with variable parameters in a finite interval of time. In the considered methods there is used transformation of an equation of free oscillations into a system of equations relative to canonical components of its solution.

Systems of equations relative to canonical components possess the following very important property: they belong to class K. This directly follows from properties of coefficients of these systems shown in Chapter II. Since these systems belong to class K, it allows us to apply to them all the constructions and conclusions, given in the preceding chapter, of the theory of systems of linear, uniform differential equations. Revealed with the help of such constructions, the properties of solutions of these systems, to a known measure, determine the properties of solutions of an equation of free oscillations; in particular, this pertains to the behavior of solutions in a finite interval of time. The degree of fulness of cognition of the properties of the latter essentially depends on the properties of coefficients of systems of equations relative to canonical components and it is higher the "nearer" these systems are to systems with matrices of coefficients of diagonal or triangular form. Thus, the effectiveness of results delivered by methods considered below, on the one hand, depends on these methods and, on the other hand, on the form and quantitative characteristics of the selected canonical expansion of solution of equation of oscillations.

§ 1. Majorants and Minorants of Real Particular Solution of Equation of Oscillations

In §§ 10 and 11 of the preceding chapter were found sequences of majorants and minorants of norm and phase coefficients of particular solution of system of linear, uniform differential equations of class K of the form (3.1):

$$\dot{x}_i = a_{i1}x_1 + \ldots + a_{in}x_n \quad (i = 1, \ldots, n). \tag{4.1}$$

Determining coefficients a_{ij} of this system by formulas

$$a_{ij} = \lambda_i \delta_{ij} + g_{ij} \quad (i, j = 1, \ldots, n) \tag{4.2}$$

or

$$a_{ij} = \zeta_i^{(l-1)} \delta_{ij} + h_{ij}^{(l)} \quad (i, j = 1, \ldots, n), \tag{4.3}$$

where δ_{ij} is Kronecker delta, we will obtain as particular cases of system (4.1), a system of equations relative to canonical components of solution of equation (0.1) $y_1, \ldots, y_n$ or a system of equations relative to canonical components $z_1, \ldots, z_n$ and, as particular cases of above-indicated sequences, sequences of majorants and minorants of norm and phase coefficients of corresponding system of differential equations relative to canonical components.

On the basis of first equation of system (2.27), solution of equation (0.1) $x(t)$ is connected with canonical components $y_1(t), \ldots, y_n(t)$ by dependence

$$x(t) = \sum_{j=1}^{n} y_j(t).$$

Analogous dependence connects solution $x(t)$ with canonical components $z_1(t), \ldots, z_n(t)$.

Using these dependences, formulas (2.34), (2.43), (2.47) or (2.49) and majorant and minorant appraisals for functions $f_j(t)$ and $r(t)$, we will establish majorant and minorant appraisals of real particular solution $x(t)$ of equation (0.1), assuming that initial data of this solution is known.

Let us consider two methods of construction of such appraisals.

1. Because of formulas

$$x(t) = \sum_{i=1}^{n} y_i(t) \quad \text{or} \quad x(t) = \sum_{i=1}^{n} z_i(t),$$

(3.4) and (3.53) we will obtain

$$x(t) = r\left(\sum_{i=1}^{n-2m} f_i + 2\sum_{l=1}^{m} f_{n-2m+2l-1}\right). \tag{4.4}$$

From equality (4.4) follows

$$\left.\begin{aligned}
\operatorname{Min} x(t) = \min\Big[&\operatorname{Min}_m r \sum_{i=1}^{n-2m} \max(\operatorname{Min}_k f_i, -1),\\
&\operatorname{Maj}_m r \sum_{i=1}^{n-2m} \max(\operatorname{Min}_k f_i, -1)\Big] +\\
+\sqrt{2}\min\Big[&\operatorname{Min}_m r \sum_{i=1}^{m} \max(\operatorname{Min}_k f_{n-2m+2i-1}, -1),\\
&\operatorname{Maj}_m r \sum_{i=1}^{n-2m} \max(\operatorname{Min}_k f_{n-2m+2i-1}, -1)\Big];\\
\operatorname{Maj} x(t) = \max\Big[&\operatorname{Min}_m r \sum_{i=1}^{n-2m} \min(\operatorname{Maj}_k f_i, 1),\\
&\operatorname{Maj}_m r \sum_{i=1}^{n-2m} \min(\operatorname{Maj}_k f_i, 1)\Big] +\\
+\sqrt{2}\max\Big[&\operatorname{Min}_m r \sum_{i=1}^{m} \min(\operatorname{Maj}_k f_{n-2m+2i-1}, 1),\\
&\operatorname{Maj}_m r \sum_{i=1}^{n-2m} \max(\operatorname{Min}_k f_{n-2m+2i-1}, 1)\Big].
\end{aligned}\right\} \tag{4.5}$$

Equalities (4.5) determine a whole class of majorants and minorants of particular solution of equation of oscillations. Considering majorants and minorants of functions $f_i(t)$ $(i = 1, \ldots, n)$ of first, second, ..., k-th approximation and majorants and minorants of function $r(t)$ of first, second, ..., m-th approximation, considering $m = k$ or $m = k + 1$ and increasing k, we will obtain sequences of majorants and minorants. These sequences converge to solution during $k \to \infty$.

Accuracy of appraisals (4.5) drops with growth of t. At sufficiently large value of t there can appear more effective (at the same value of m) appraisal

$$\left.\begin{aligned}
\operatorname{Min} x(t) &= -\sqrt{n}\operatorname{Maj}_m r,\\
\operatorname{Maj} x(t) &= \sqrt{n}\operatorname{Maj}_m r.
\end{aligned}\right\} \tag{4.6}$$

Using the result of inequality

$$x\bar{x} = \begin{cases} = (y_1 + \ldots + y_n)(\bar{y}_1 + \ldots + \bar{y}_n) \leqslant nr^2,\\ = (z_1 + \ldots + z_n)(\bar{z}_1 + \ldots + \bar{z}_n) \leqslant nr^2. \end{cases} \tag{4.7}$$

In those cases when this will take place, appraisals (4.5) (one or both) one should replace by appraisals (4.6).

2. Summarizing separately left and right side of equations (4.1) and considering $x_i = y_i$ in case (4.2) or $x_i = z_i$ in case (4.3), we will find

$$\dot{x}=\begin{cases}-\sum_{i,j=1}^{n}(\lambda_i\delta_{ij}+g_{ij})y_j\\ -\sum_{i,j=1}^{n}(\zeta_i^{(l-1)}\delta_{ij}+h_{ij}^{(l)})z_j.\end{cases}\tag{4.8}$$

Majorant and minorant appraisals of first approximation for function x(t) we will construct in the following form.

Expressing in equation (4.1) canonical components y_i (or z_i) through norm of solution r and phase coefficients $e_1, \ldots, e_n$, we will obtain

$$\dot{x}=\begin{cases}-r\sum_{i,j=1}^{n}(\lambda_i\delta_{ij}+g_{ij})e_j\\ -r\sum_{i,j=1}^{n}(\zeta_i^{(l)}\delta_{ij}+h_{ij}^{(l+1)})e_j\\ (l=0,1,2,\ldots)\end{cases}\tag{4.9}$$

After designating magnitude

$$\sqrt{\sum_{j=1}^{n}\left[\sum_{i=1}^{n}(\lambda_i\delta_{ij}+g_{ij})\sum_{i=1}^{n}(\bar{\lambda}_i\delta_{ij}+\bar{g}_{ij})\right]}$$

or

$$\sqrt{\sum_{j=1}^{n}\left[\sum_{i=1}^{n}(\zeta_i^{(l)}\delta_{ij}+h_{ij}^{(l+1)})\sum_{i=1}^{n}(\bar{\zeta}_i^{(l)}\delta_{ij}+\bar{h}_{ij}^{(l+1)})\right]}$$

by symbol L, on the basis of known algebraic inequality, and taking into account equation (4.9), we will obtain

$$|\dot{x}|<rL\sqrt{\sum_{j=1}^{n}e_j\bar{e}_j}.$$

Because of conditions (3.6), this inequality takes the form

$$|\dot{x}|<Lr.\tag{4.10}$$

Applying majorant appraisal of function r(t), S(t) (see § 7 of preceding chapter), on the basis of inequality (4.10) we will find

$$\left.\begin{aligned}\mathrm{Min}_1\,x(t)&=x(t_0)-\int_{t_0}^{t}L(t)S(t)\,dt=\\ &=x(t_0)-r(t_0)\int_{t_0}^{t}L(t)\exp\int_{t_0}^{t}\mu_n(\tau)\,d\tau\,dt,\\ \mathrm{Maj}_1\,x(t)&=x(t_0)+\int_{t_0}^{t}L(t)S(t)\,dt=\\ &=x(t_0)+r(t_0)\int_{t_0}^{t}L(t)\exp\int_{t_0}^{t}\mu_n(\tau)\,d\tau\,dt.\end{aligned}\right|\tag{4.11}$$

Effectiveness of appraisals (4.11) is higher the less t. During sufficiently large t there can appear more effective appraisals

$$\left.\begin{aligned}\mathrm{Min}_1 x(t) &= -\sqrt{n}\, r(t_0)\exp\int_{t_0}^{t}\mu_n(\tau)\,d\tau,\\ \mathrm{Maj}_1 x(t) &= \sqrt{n}\, r(t_0)\exp\int_{t_0}^{t}\mu_n(\tau)\,d\tau,\end{aligned}\right\} \tag{4.12}$$

being particular case of appraisals (4.6) ($m = 1$).

Majorant and minorant appraisals of k-th approximation we will construct also by equations (4.9). With this goal, we will cross from variables $e_1, \ldots, e_n$ to variables $f_1, \ldots, f_n$ and will copy equation (4.9) in the form

$$\dot{x} = r\sum_{j=1}^{n} q_j(t) f_j, \tag{4.13}$$

where

$$q_j = \sum_{i=1}^{n} a_{ij} \quad \text{— at } j \leq n-2m;$$

$$q_j = \sqrt{2}\,\mathrm{Re}\sum_{i=1}^{n} a_{ij} \quad \text{— at } j = n-2m+1, \ldots, n-1;$$

$$q_j = \sqrt{2}\,\mathrm{Im}\sum_{i=1}^{n} a_{ij} \quad \text{— at } j = n-2m+2, \ldots, n;$$

and coefficients a_{ij} are determined by formulas (4.2) or (4.3).

Designating magnitude

$$\min\sum_{j=1}^{n} q_j(t) f_j,$$

found during conditions (3.54) and

$$\mathrm{Min}_{k-1} f_j \leq f_j \leq \mathrm{Maj}_{k-1} f_j \qquad (j = 1, \ldots, n).$$

by symbol $\chi_k(t)$ and magnitude

$$\max\sum_{j=1}^{n} q_j(t) f_j,$$

found with those same conditions, by symbol $\xi_k(t)$, because of equation (4.13) we will obtain

$$r(t)\chi_k(t) \leq \dot{x}(t) \leq \xi_k(t) r(t). \tag{4.14}$$

Estimating function $r(t)$ by minorant and majorant of k-th approximation, we will obtain the following formulas for minorants and majorants of k-th approximation of function $x(t)$:

$$\left.\begin{aligned}\mathrm{Min}_k x(t) &= x(t_0) + \int_{t_0}^{t}\min[\chi_k(t)\,\mathrm{Min}_k r(t),\\ &\qquad \chi_k(t)\,\mathrm{Maj}_k r(t)]\,dt;\\ \mathrm{Maj}_k x(t) &= x(t_0) + \int_{t_0}^{t}\max[\xi_k(t)\,\mathrm{Min}_k r(t),\\ &\qquad \xi_k(t)\,\mathrm{Maj}_k r(t)]\,dt.\end{aligned}\right\} \tag{4.15}$$

During sufficiently large values of t these appraisals can appear less exact than appraisals delivered by formulas (4.6) in which is assumed $m = k$. In those cases when this will take place, appraisals (4.15), one or both, one should replace by appraisals (4.6).

It is not difficult to show that sequences, built by first or second method, of majorants and minorants of function x(t) possess the following properties:

a) $\mathrm{Min}_1 x(t) \leq \mathrm{Min}_2 x(t) \leq \ldots \leq x(t) \leq \ldots \leq \mathrm{Maj}_2 x(t) \leq \mathrm{Maj}_1 x(t)$;

b) in any finite interval (t_0, T)

$$\lim_{k\to\infty} \mathrm{Min}_k x(t) = \lim_{k\to\infty} \mathrm{Maj}_k x(t) = x(t).$$

Example: In § 7 of the preceding chapter there is given a system of equations (3.49) obtained as a result of canonical expansion of the first form of solution of equation (3.48)

$$\ddot{x} - (t^2 + a)x = 0. \tag{4.16}$$

Coefficients of this system λ_1, λ_2, g_{11}, g_{12}, g_{21}, g_{22} have the form

$$\lambda_1 = -\lambda_2 = -\sqrt{t^2 + a},$$

$$g_{11} = -g_{12} = -g_{21} = g_{22} = -\frac{t}{2(t^2 + a)}.$$

During initial conditions

$$x(0) = 1, \ \dot{x}(0) = -1$$

in case a = 1 we will obtain

$$e_1(0) = f_1(0) = 1, \ e_2(0) = f_2(0) = 0, \ r(0) = 1.$$

In §§ 10 and 11 of the preceding chapter, with these initial conditions for case a = 1 in section [0.1], there are built majorants and minorants of the first approximation of functions $f_1(t)$ and $f_2(t)$ and majorants and minorants of the first and second approximations of function r(t). Using these results and considering in equation (4.16) a = 1, we will find majorants and minorants of the first and second approximations of considered particular solution of equation (4.16) by both presented methods.

For determination of majorants and minorants of first approximation by the first method we will use formulas (4.5), assuming

$$m = k = 1.$$

In this case these formulas will take the form

$$\mathrm{Min}_1\, x(t) = \min\left[\mathrm{Min}_1\, r \sum_{i=1}^{2} \max(\mathrm{Min}_1 f_i, -1),\right.$$
$$\left.\mathrm{Maj}_1\, r \sum_{i=1}^{2} \max(\mathrm{Min}_1 f_i, -1)\right];$$
$$\mathrm{Maj}_1\, x(t) = \max\left[\mathrm{Min}_1\, r \sum_{i=1}^{2} \min(\mathrm{Maj}_1 f_i, 1),\right.$$
$$\left.\mathrm{Maj}_1\, r \sum_{i=1}^{2} \min(\mathrm{Maj}_1 f_i, 1)\right].$$

For determination of majorants and minorants of second approximation by the same method we will apply formulas (4.5), assuming

$$m=2,\ k=1.$$

We will obtain

$$\mathrm{Min}_2\, x(t) = \min\left[\mathrm{Min}_2\, r \sum_{i=1}^{2} \max(\mathrm{Min}_1 f_i, -1),\right.$$
$$\left.\mathrm{Maj}_2\, r \sum_{i=1}^{2} \max(\mathrm{Min}_1 f_i, -1)\right];$$
$$\mathrm{Maj}_2\, x(t) = \max\left[\mathrm{Min}_2\, r \sum_{i=1}^{2} \min(\mathrm{Maj}_1 f_i, 1),\right.$$
$$\left.\mathrm{Maj}_2\, r \sum_{i=1}^{2} \min(\mathrm{Maj}_1 f_i, 1)\right].$$

Results of calculations by given formulas are presented in the form of graphs on Fig. 10.

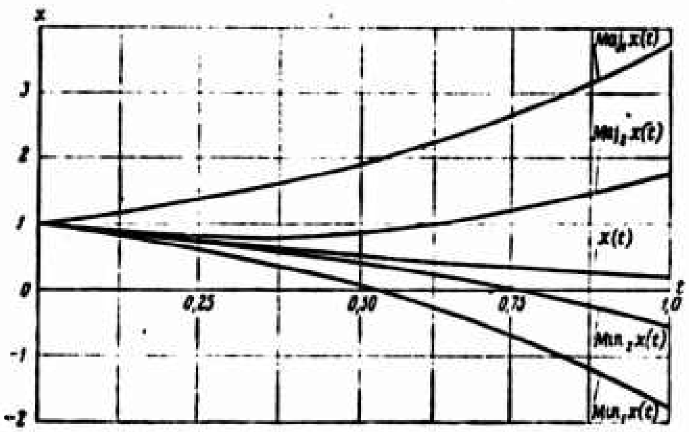

Fig. 10. Majorants and minorants of particular solution of equation $\ddot{x} - (t^2 + 1)x = 0$.

For determination of majorants and minorants of first approximation by the second method we will find, preliminarily, function

$$L(t) = \sqrt{\lambda_1^2 + \lambda_2^2}.$$

In accordance with formulas for roots λ_1 and λ_2 we have

$$L(t) = \sqrt{2(t^2+2)}.$$

Functions $\mathrm{Min}_1 x(t)$ and $\mathrm{Maj}_1 x(t)$ we will find by formulas (4.11).

For construction of $\mathrm{Min}_2 x(t)$ and $\mathrm{Maj}_2 x(t)$ we will find, preliminarily, auxiliary functions $\chi_2(t)$ and $\xi_2(t)$. For considered equation we have

$$\left.\begin{aligned}\chi_2(t) &= \min(\lambda_1 f_1 + \lambda_2 f_2) = -\sqrt{t^2+1}\max(f_1 - f_2),\\ \xi_2(t) &= \max(\lambda_1 f_1 + \lambda_2 f_2) = \sqrt{t^2+1}\max(f_2 - f_1)\end{aligned}\right\}$$

during conditions

$$\left.\begin{aligned}&f_1^2 + f_2^2 = 1,\\ &\mathrm{Min}_1 f_1 < f_1 < \mathrm{Maj}_1 f_1,\\ &\mathrm{Min}_1 f_2 < f_2 < \mathrm{Maj}_1 f_2.\end{aligned}\right\}$$

At $t \geq 0.4$ difference $f_1 - f_2$ attains maximum when $f_1 = 0.70$ and $f_2 = 0.70$ and these two conditions do not limit magnitude of maximum (see Fig. 7).

At $t \leq 0.3$ difference $f_1 - f_2$ is limited from above by inequality

$$f_2 > \mathrm{Min}_1 f_2$$

and attains maximum when $f_2 = f_2^{(0)}$, $f_1 = f_1^{(0)}$, where

$$\begin{aligned}f_2^{(0)} &= \mathrm{Min}_1 f_2,\\ f_1^{(0)} &= \sqrt{1-(\mathrm{Min}_1 f_2)^2}.\end{aligned}$$

Difference $f_2 - f_1$ during $t \geq 0.8$ attains maximum when $f_2 = 0.7$ and $f_1 = -0.7$. During $t \leq 0.7$ conditions of its maximum are affected by fixed limitations of possible values of magnitudes f_1 and f_2 (see Fig. 7); at $0 \leq t \leq 0.4$ difference $f_2 - f_1$ attains maximum when $f_1 = f_1^{(1)}$, $f_2 = f_2^{(1)}$, where

$$\begin{aligned}f_1^{(1)} &= \mathrm{Min}_1 f_1,\\ f_2^{(1)} &= \sqrt{1-(\mathrm{Min}_1 f_1)^2};\end{aligned}$$

at $0.5 \leq t \leq 0.7$ difference $f_2 - f_1$ attains maximum, when $f_2 = f_2^{(2)}$, $f_1 = f_1^{(2)}$, where $f_2^{(2)} = \mathrm{Maj}_1 f_2$;

$$f_1^{(2)} = \sqrt{1-(\mathrm{Maj}_1 f_2)^2}.$$

Value of functions $\chi_2(t)$ and $\xi_2(t)$ at t = 0; 0.1; 0.2; ...; 1 are given in Table 2.

Table 2.

t	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1,0
χ_2	−1,00	−1,18	−1,35	−1,46	−1,52	−1,58	−1,65	−1,72	−1,80	−1,90	−1,99
ξ_2	−1,00	−0,18	−0,51	−0,20	0,27	1,18	1,45	1,67	1,90	1,90	1,99

Considering data of calculation of functions $\chi_2(t)$ and $\xi_2(t)$, $\mathrm{Min}_2 r(t)$ and $\mathrm{Maj}_2 r(t)$ (see Fig. 8) by formula (4.15) for k = 2 we will find function $\mathrm{Min}_2 x(t)$ and

$\mathrm{Maj}_2x(t)$. Graphs of these functions, and also of earlier found functions $\mathrm{Min}_1x(t)$ and $\mathrm{Maj}_1x(t)$, are found on Fig. 10.

On Fig. 10 are depicted graphs of function $x(t)$, for which are built appraisals found above. Function $x(t)$ is calculated for considered initial conditions by the formula of general solution given in § 7 of the preceding chapter.

§ 2. Approximate Presentations of General Solution of Equation of Oscillations

Using canonical expansions and assuming that coefficients of equation of oscillations are differentiable a sufficient number of times, we will construct approximate presentations of its general solution.

In order to find one such presentation, one should assume that l-th canonical expansion of solution of equation of oscillations will convert this equation into such a system of equations relative to canonical components whose matrix of coefficients is diagonal (i.e., at $j \neq k$ it does not have coefficients $h_{jk}^{(l)}$).

As formula for coefficients $h_{jk}^{(l)}$ shows, from condition $h_{jk}^{(l)} = 0$ during $j \neq k$ there follows $h_{jk}^{(l)} = 0$ during $j = k$. Therefore, having made the above-indicated assumption, we will obtain a system of equations relative to canonical components in the form

$$\dot{z}_j = \zeta_j^{(l-1)} z_j \quad (j = 1, \ldots, n). \tag{4.17}$$

Solution of system (4.17) has the form

$$z_j = C_j \exp \int \zeta_j^{(l-1)} dt \quad (j = 1, \ldots, n), \tag{4.18}$$

where C_j ($j = 1, \ldots, n$) are arbitrary constants (in general, complex).

In accordance with equations (4.18), general solution of equation (0.1) we will obtain in the form

$$x(t) = C_1 \exp \int \zeta_1^{(l-1)} dt + \ldots + C_n \exp \int \zeta_n^{(l-1)} dt. \tag{4.19}$$

If matrix of coefficients of system of equations relative to canonical components is indeed diagonal, equation (4.19) will be executed with the sign of strict equality and will be an exact formula of general solution of equation (0.1). If mentioned condition is not executed, sign of strict equality in equation (4.19) one should replace by sign of approximate equality: in this case it is possible to consider

right side of equation (4.19) as approximate presentation $\tilde{x}(t)$ of general solution of equation (0.1):

$$\tilde{x}(t) = C_1 \exp \int \zeta_1^{(l-1)} dt + \ldots + C_n \exp \int \zeta_n^{(l-1)} dt. \tag{4.20}$$

Another form of approximate presentation of general solution can be obtained if one were to be limited by assumption about diagonal form of matrix of coefficients of a system relative to canonical components but not use condition "$h_{jk} = 0$ during $j = k$, if $h_{jk} = 0$ during $j \neq k$." In this case, instead of formula (4.20), we will obtain

$$\begin{aligned} \tilde{x}(t) = C_1 \exp \int (\zeta_1^{(l-1)} + h_{11}^{(l)}) dt + \ldots \\ \ldots + C_n \exp \int (\zeta_n^{(l-1)} + h_{nn}^{(l)}) dt. \end{aligned} \tag{4.21}$$

Formula (4.21) for a broad class of equations gives closer coincidence of approximate presentation of solution with true solution [under the condition that in formulas (4.20) and (4.21) numbers l are identical].

Function $\tilde{x}(t)$, determined by formulas (4.20) or (4.21), it is possible to present in the form

$$\tilde{x}(t) = \sum_{j=1}^{n} \tilde{x}_j(t). \tag{4.22}$$

where

$$\tilde{x}_j(t) = C_j \exp \int \zeta_j^{(l-1)} dt \tag{4.23}$$

in the case of formula (4.20) and

$$\tilde{x}_j(t) = C_j \exp \int (\zeta_j^{(l-1)} + h_{jj}^{(l)}) dt \tag{4.24}$$

in the case of formula (4.21).

We will estimate accuracy of approximate presentation of general solution (4.20).

Let us assume that particular solution is given of equation (0.1) by initial conditions (2.2). Then, t_0 the lower limit of integrations, constants C_j one can determine from system of equations

$$\left.\begin{aligned} & C_1 + \ldots + C_n = \xi_0, \\ & \zeta_1^{(l-1)}(t_0) C_1 + \ldots + \zeta_n^{(l-1)}(t_0) C_n = \xi_1, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & [(\zeta_1^{(l-1)}(t_0) + D)^{n-2} \zeta_1^{(l-1)}(t)]_{t=t_0} C_1 + \ldots + [(\zeta_n^{(l-1)}(t_0) + D)^{n-2} \times \\ & \times \zeta_n^{(l-1)}(t)]_{t=t_0} C_n = \xi_{n-1}. \end{aligned}\right\}$$

Found values of arbitrary constant during known functions $\zeta_1^{(l-1)}, \ldots, \zeta_n^{(l-1)}$ simply determine components $\tilde{x}_j(t)$ of approximate solution. With this, to every component $\tilde{x}_j(t)$ correspond initial conditions

$$\left.\begin{aligned} &\tilde{x}_j(t_0) = C_j, \\ &\left(\frac{d\tilde{x}_j}{dt}\right)_{t=t_0} = \zeta_j^{(l-1)}(t_0)\,C_j, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\left(\frac{d^{n-1}\tilde{x}_j}{dt^{n-1}}\right)_{t=t_0} = [(\zeta_j^{(l-1)}(t) + D)\,\zeta_j^{(l-1)}(t)]_{t=t_0} C_j. \end{aligned}\right| \tag{4.25}$$

Substitution of function $\tilde{x}_j(t)$ instead of function $x(t)$ in the left part of equation (0.1) will give

$$\frac{d^n\tilde{x}_j}{dt^n} + b_1\frac{d^{n-1}\tilde{x}_j}{dt^{n-1}} + \ldots + b_n\tilde{x}_j = -\vartheta_j\tilde{x}_j, \tag{4.26}$$

where

$$\vartheta_j = [(\zeta_j^{(l-1)} + D)^{n-1}\zeta_j^{(l-1)}] + b_1[(\zeta_j^{(l-1)} + D)^{n-1}\zeta_j^{(l-1)}] + \ldots$$
$$\ldots + b_n = -\frac{\nabla_l}{\pi_{nj}^{(l)}}\,h_{jj}^{(l)}. \tag{4.27}$$

Consequently, function $\tilde{x}_j(t)$ is solution of equation

$$\frac{d^n\tilde{x}_j}{dt^n} + b_1\frac{d^{n-1}\tilde{x}_j}{dt^{n-1}} + \ldots + b_{n-1}\frac{d\tilde{x}_j}{dt} + (b_n + \vartheta_j)\,\tilde{x} = 0, \tag{4.28}$$

in which coefficient $b_n + \vartheta_j$, in general is a complex-valued function of t.

Now we will present true solution of equation (0.1) in the form of a sum with the help of equality

$$x(t) = \sum_{j=1}^{n} x_j(t), \tag{4.29}$$

where $x_j(t)$ are particular solutions of equation (0.1), coinciding with corresponding approximate solutions $\tilde{x}_j(t)$ in initial conditions.

For appriasal of error of approximation we will introduce variables $u_1, \ldots, u_n$ by equalities

$$u_j = \tilde{x}_j - x_j \quad (j = 1, \ldots, n). \tag{4.30}$$

Subtracting from equation (4.28), equation

$$\frac{d^n x_j}{dt^n} + b_1\frac{d^{n-1}x_j}{dt^{n-1}} + \ldots + b_n x_j = 0, \tag{4.31}$$

we will find

$$\frac{d^n u_j}{dt^n}+b_1\frac{d^{n-1}u_j}{dt^{n-1}}+\ldots+b_n u_j=-b_j\tilde{x}_j. \tag{4.32}$$

Consequently, magnitude u_j during arbitrary value of index j satisfies equation (4.32). Right side of this equation we know, inasmuch as approximate solution is known; initial conditions for variable u_j, because of construction of functions $\tilde{x}_j(t)$ and $x_j(t)$, are zero.

Expanding solution of system (4.32) by transformation

$$\left.\begin{aligned}
&u_j=v_{j1}+\ldots+v_{jn},\\
&\frac{du_j}{dt}=\zeta_1^{(l-1)}v_{j1}+\ldots+\zeta_n^{(l-1)}v_{jn},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n-1}u_j}{dt^{n-1}}=[(\zeta_1^{(l-1)}+D)^{n-2}\zeta_1^{(l-1)}]v_{j1}+\ldots\\
&\qquad\ldots+[(\zeta_n^{(l-1)}+D)^{n-2}\zeta_n^{(l-1)}]v_{jn}
\end{aligned}\right| \tag{4.33}$$

to canonical components $v_{j1}, \ldots, v_{jn}$ and excluding from equations (4.33) and (4.32) variable u_j (as this was done in § 4, Chapter II), we will obtain a system of equations relative to canonical components of error $v_{j1}, \ldots, v_{jn}$ in the form

$$\dot{v}_{jr}=\zeta_r^{(l-1)}v_{jr}+\sum_{s=1}^{n}h_{rs}^{(l)}v_{js}-\frac{b_j\tilde{x}_j w_{nr}^{(l)}}{W_l}\;(r=1,\ldots,n). \tag{4.34}$$

Passing in equations (4.34), to conjugate complex numbers, multiplying r-th equation of system (4.34) by $\bar{v}_{jr}$ and r-th equation of new system by v_{jr}, and term by term adding all equations, we will obtain

$$\begin{aligned}
2\dot{r}_j r_j=r_j^2\sum_{r}^{n}\Big[(\zeta_r^{(l-1)}+\bar{\zeta}_r^{(l-1)})e_{jr}\bar{e}_{jr}+\\
+\bar{e}_{jr}\sum_{s=1}^{n}h_{rs}^{(l)}e_{js}+e_{jr}\sum_{s=1}^{n}\bar{h}_{rs}^{(l)}\bar{e}_{js}\Big]-\\
-\frac{b_j\tilde{x}_j r_j}{W_l}\sum_{r=1}^{n}w_{nr}^{(l)}\bar{e}_{jr}-\frac{\bar{b}_j\tilde{x}_j r_j}{\bar{W}_l}\sum_{r=1}^{n}\bar{w}_{nr}^{(l)}e_{jr},
\end{aligned} \tag{4.35}$$

where

$$r_j=\sqrt{v_{j1}\bar{v}_{j1}+\ldots+v_{jn}\bar{v}_{jn}}; \tag{4.36}$$

$$e_{jr}=\frac{v_{jr}}{r_j}\quad(r=1,\ldots,n). \tag{4.37}$$

From equation (4.35) follows

$$\dot{r}_j<r_j\mu_n+\left|\frac{b_j\tilde{x}_j}{W_l}\right|\sqrt{\sum_{r=1}^{n}w_{nr}^{(l)}\bar{w}_{nr}^{(l)}}. \tag{4.38}$$

where

$$\mu_n = \max \cdot \frac{1}{2} \sum_{\nu=1}^{n} [(\zeta_\nu^{(\nu-1)} + \bar{\zeta}_\nu^{(\nu-1)}) e_{j\nu} \bar{e}_{j\nu} + \bar{e}_{j\nu} \sum_{s=1}^{n} h_{\nu s}^{(1)} e_{js} + e_{j\nu} \sum_{s=1}^{n} \bar{h}_{\nu s}^{(1)} \bar{e}_{js}]$$

during condition

$$|e_{j1}|^2 + \ldots + |e_{jn}|^2 = 1.$$

Applying formula of solution of linear, first order differential equation and considering condition $r(t_0) = 0$, because of inequality (4.38) we will obtain

$$r_j \leqslant \exp\left(\int_{t_0}^{t} \mu_n\, dt\right)\left[\int_{t_0}^{t} \left|\frac{b_j \tilde{x}_j}{W_t}\right| \sqrt{\sum_{\nu=1}^{n} w_{n\nu}^{(1)} \bar{w}_{n\nu}^{(1)}} \exp\left(-\int_{t_0}^{t} \mu_n\, dt\right) dt\right]. \tag{4.39}$$

But since

$$|u_j| \leqslant \sqrt{n}\, r_j$$

(see preceding paragraph), from inequality (4.39) follows

$$|\tilde{x}_j - x_j| \leqslant \sqrt{n} \exp\left(\int_{t_0}^{t} \mu_n\, dt\right)\left[\int_{t_0}^{t} \left|\frac{b_j \tilde{x}_j}{W_t}\right| \times \sqrt{\sum_{\nu=1}^{n} w_{n\nu}^{(1)} \bar{w}_{n\nu}^{(1)}} \exp\left(-\int_{t_0}^{t} \mu_n\, dt\right) dt\right]. \tag{4.40}$$

Considering j = 1, ..., n, we will obtain a system of inequalities, with the help of which it is possible to estimate accuracy of approximate solution found by formula (4.20) during any given initial conditions.

Analogously, it is possible to estimate accuracy of approximate presentation of general solution (4.21).

<u>Example</u>: For equation

$$\frac{d^2x}{dt^2} + ct^\nu x = 0, \quad c < 0, \quad \nu > -2 \tag{4.41}$$

roots $\lambda_{1,2}$ during $t > 0$ are different and have the form

$$\lambda_{1,2} = \mp t^{\nu/2} \sqrt{-c}.$$

Considering interval $(0, \infty)$ and applying first canonical expansion, we will obtain, for coefficients of system (2.43), expression

$$h_{11}^{(1)} = h_{22}^{(1)} = -h_{21}^{(1)} = -h_{12}^{(1)} = -\frac{\nu}{4t}.$$

During

$$t^{\nu+2} > \frac{\nu^2}{16c} \tag{4.42}$$

roots of equation (2.45) are different and are expressed by formula

$$\varkappa^{(1)}_{1,2} = -\frac{\nu}{4t} \mp \frac{1}{2}\sqrt{\frac{\nu^2}{4t^2} - 4ct^{\nu}}.$$

Assuming that oscillation are considered in interval of time for which condition (4.42) is executed, we will find coefficients $h^{(2)}_{ij}$ (i, j = 1, 2) of system (2.47), obtained as a result of second cancnical expansion of solution of equation (0.1).

We will obtain

$$h^{(2)}_{11} = -h^{(2)}_{21} = +\frac{\nu}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-\frac{1}{2}} + \frac{\nu}{4t} - \frac{1}{4}\frac{d\ln\theta}{dt},$$

$$h^{(2)}_{22} = -h^{(2)}_{12} = -\frac{\nu}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-\frac{1}{2}} + \frac{\nu}{4t} - \frac{1}{4}\frac{d\ln\theta}{dt},$$

where

$$\theta = \frac{\nu^2}{4t^2} - 4ct^{\nu}.$$

Applying formula (4.21) during l = 2, we will obtain approximate presentation of solution $\tilde{x}(t)$ in the form

$$\tilde{x}(t) = \theta^{-\frac{1}{4}}\left\{C_1\exp\int\theta^{-\frac{1}{2}}\left(2ct^{\nu} + \frac{\nu}{4t^2}\right)dt + C_2\exp\int\left[-\theta^{-\frac{1}{2}}\left(2ct^{\nu} + \frac{\nu}{4t^2}\right)\right]dt\right\}. \tag{4.43}$$

In particular, for c = 1, ν = 1 this formula will take the form

$$\tilde{x}(t) = C_1'\left(t - \frac{1}{16t^2}\right)^{-\frac{1}{4}}\sin\left\{\frac{1}{6}\sqrt{16t^3-1} - \frac{1}{3}\arcsin\frac{1}{4\sqrt{t^3}} + C_2'\right\}. \tag{4.44}$$

As is known, solution of equation (4.41) during c = 1 and ν = 1 is expressed through Bessel functions and for initial conditions $x(0) = 1$, $x(0) = 0$ has the form [35]

$$x(t) = \frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}}\sqrt{t}\,J_{-\frac{1}{3}}\left(\frac{2}{3}t^{\frac{3}{2}}\right). \tag{4.45}$$

Graph of this solution for interval (0.10) is shown in Fig. 11. Derivative of solution (4.45) has the form [35]

$$\dot{x}(t) = -\sqrt[3]{3}\,\Gamma\left(\frac{4}{3}\right)\sqrt{t}\,J_{\frac{1}{3}}\left(\frac{2}{3}t^{\frac{3}{2}}\right). \tag{4.46}$$

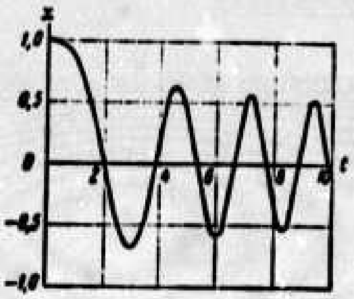

Fig. 11. Particular solution of equation $\ddot{x} + tx = 0$.

During t = 2, according to formulas (4.45) and (4.46), we have

$$\left.\begin{aligned} x &= -0{,}015, \\ \dot{x} &= -1{,}097. \end{aligned}\right\} \qquad (4.47)$$

At t > 2 condition (4.42) is executed and, consequently, formula (4.44) of approximate presentation is applicable.

Calculation of solution by formula (4.44) during initial conditions (4.47) in interval (2.10) showed that divergence between approximate and exact solutions does not exceed 0.01. Data of approximate and exact calculations are given in Table 3.

Table 3.

t	2,0	2,25	2,5	2,75	3,0	3,25	3,5	3,75
x exact	−0,015	−0,282	−0,510	−0,658	−0,695	−0,603	−0,392	−0,096
x approximate	−0,015	−0,280	−0,506	−0,651	−0,686	−0,594	−0,384	−0,093

t	4,0	4,25	4,5	4,75	5,0	5,25	5,5	5,75
x exact	0,220	0,482	0,618	0,584	0,382	0,063	−0,271	−0,519
x approximate	0,219	0,477	0,610	0,575	0,371	0,059	−0,272	−0,513

t	6,0	6,25	6,5	6,75	7,0	7,25	7,5	7,75
x exact	−0,582	−0,435	−0,123	0,236	0,498	0,550	0,362	0,011
x approximate	−0,575	−0,426	−0,118	0,235	0,492	0,541	0,354	0,004

t	8,0	8,25	8,5	8,75	9,00	9,25	9,5	9,75	10,0
x exact	−0,344	−0,533	−0,459	−0,152	0,233	0,492	0,480	0,197	−0,199
x approximate	−0,340	−0,525	−0,451	−0,147	0,232	0,486	0,472	0,194	−0,198

§ 3. Weighted Canonical Components and Systems of Equations Which They Satisfy

Presented in § 1, methods of construction of majorants and minorants of a particular solution of an equation of free oscillations, it would be possible to consider, in a certain measure, complete if, during application to equations with constant coefficients in any stage of successive approximations, they led to the determination of majorants and minorants which coincide with the considered particular solution. Unfortunately, the presented methods do not possess such a property. The only exceptions are such cases of determination of majorants and minorants of particular solutions of equations with constant coefficients, in which during $t = t_0$ magnitude $\dot{r}/r$ takes maximum possible value, equal to one of the real roots of the characteristic equation. In these cases certain phase coefficient $f_1(=e_1)$ is numerically equal to unity. During positive f_1, majorant of first approximation of investigated particular solution $x(t)$, built by first method, coincides with solution; analogously, during negative coefficient f_1, with solution coincides its minorant of first approximation (with the same method of construction).

Shown deficiency of considered methods may be removed by means of a certain modification of them, in basis of which are placed the found general solution of equation of oscillations.

Let us assume that function $\zeta_1^{(k-1)}(t), \ldots, \zeta_n^{(k-1)}(t)$ satisfies conditions of applicability of k-th canonical expansion of second form, and we will introduce variables $Z_1, \ldots, Z_n$,

$$Z_i = \frac{c_i z_i}{\tilde{x}_i} \quad (i = 1, \ldots, n), \tag{4.48}$$

where under $\tilde{x}_i$ we will understand either approximate solution of equation of oscillations of form (4.23) [corresponding to formula (4.20) of approximate presentation of general solution of equation of oscillations] or approximate solution of mentioned equation of the form (4.24) [corresponding to formula (4.21) of approximate presentation of its general solution]. We will agree to call new variables <u>weighted canonical components</u>.

In formulas (4.20), (4.21), (4.23), and (4.24) there is not shown below a limit of integration. We will consider that it is the moment of time t_0, which is the beginning of that interval in which the process is considered.

After replacing in system (2.48) variables $z_1, \ldots, z_n$ by variables $Z_1, \ldots, Z_n$, using formula (4.48), we will obtain system

$$z_i \frac{\dot{\tilde{x}}_i}{C_i} + \dot{z}_i \frac{\tilde{x}_i}{C_i} = \zeta_i^{(k-1)} z_i \frac{\tilde{x}_i}{C_i} + \sum_{j=1}^{n} h_{ij}^{(k)} z_j \frac{\tilde{x}_j}{C_j}$$
$$(i = 1, \ldots, n),$$

which after elementary transformations will take the form

$$\dot{z}_i = \sum_{j=1}^{n} h_{ij}^{(k)} z_j \frac{\tilde{x}_j C_i}{\tilde{x}_i C_j} + \left(\zeta_i^{(k-1)} - \frac{\dot{\tilde{x}}_i}{\tilde{x}_i}\right) z_i \qquad (4.50)$$
$$(i = 1, \ldots, n).$$

If functions $\tilde{x}_i(t)$ ($i = 1, \ldots, n$) are particular solutions of equation of oscillations, then all coefficients h_{ij} are equal to zero and system (4.50) takes the form

$$\dot{z}_i = 0 \quad (i = 1, \ldots, n)$$

and, consequently, all weighted canonical components z_i are constant.

In the particular case when approximate solutions of equation of oscillations are determined by formulas (4.23), system (4.50) takes the form

$$\dot{z}_i = \sum_{j=1}^{n} h_{ij}^{(k)} z_j \exp \int_{t_0}^{t} \left(\zeta_j^{(k-1)} - \zeta_i^{(k-1)}\right) dt \qquad (4.51)$$
$$(i = 1, \ldots, n).$$

Analogously, formulas (4.24) of approximate solutions lead to equations

$$\dot{z}_i = \sum_{j=1, j \neq i}^{n} h_{ij}^{(k)} z_j \exp \int_{t_0}^{t} \left(\zeta_j^{(k-1)} + h_{jj}^{(k)} - \zeta_i^{(k-1)} - h_{ii}^{(k)}\right) dt \qquad (4.52)$$
$$(i = 1, \ldots, n).$$

Example: In example considered in preceding paragraph, approximate solutions $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ of equation

$$\frac{d^2x}{dt^2} + ct^{\nu}x = 0, \quad c \neq 0, \quad \nu > -2 \qquad (4.41)$$

are defined by formulas (4.24) during $l = 2$. These approximate solutions during the limit of integration (t_0), assumed below, can be recorded in the form

$$\tilde{x}_1(t) = C_1 \frac{\theta^{\frac{1}{4}}(t_0)}{\theta^{\frac{1}{4}}(t)} \exp \int_{t_0}^{t} \theta^{-\frac{1}{2}}\left(2ct^{\nu} + \frac{\nu}{4t^2}\right) dt, \quad \tilde{x}_2(t) = C_2 \frac{\theta^{\frac{1}{4}}(t_0)}{\theta^{\frac{1}{4}}(t)} \exp \int_{t_0}^{t} \left[-\theta^{-\frac{1}{2}}\left(2ct^{\nu} + \frac{\nu}{4t^2}\right)\right] dt.$$

Formulas of connection between weighted and usual canonical components (4.48) in this case take the form

$$
\left.
\begin{aligned}
Z_1 &= z_1 \cdot \frac{\theta^{\frac{1}{4}}(t)}{\theta^{\frac{1}{4}}(t_0)} \exp\left[-\int_{t_0}^{t} \theta^{-\frac{1}{2}}\left(2ct^{\nu} + \frac{\nu}{4t^2}\right)dt\right], \\
Z_2 &= z_2 \frac{\theta^{\frac{1}{4}}(t)}{\theta^{\frac{1}{4}}(t_0)} \exp \int_{t_0}^{t} \theta^{-\frac{1}{2}}\left(2ct^{\nu} + \frac{\nu}{4t^2}\right)dt.
\end{aligned}
\right\} \tag{4.53}
$$

In accordance with formulas for coefficients $\zeta_1^{(1)}$, $\zeta_2^{(1)}$, $h_{11}^{(2)}$, $h_{22}^{(2)}$, we will find

$$\zeta_2^{(1)} - \zeta_1^{(1)} + h_{22}^{(2)} - h_{11}^{(2)} = -\left(\frac{\nu}{2t^2} + 4ct^{\nu}\right)\theta^{-\frac{1}{2}}.$$

But since

$$\frac{d\theta^{\frac{1}{2}}}{dt} = -\theta^{-\frac{1}{2}}\frac{\nu}{2t}\left(\frac{\nu}{2t^2} + 4ct^{\nu}\right),$$

then from preceding equality, we will obtain

$$\zeta_2^{(1)} - \zeta_1^{(1)} + h_{22}^{(2)} - h_{11}^{(2)} = \frac{2t}{\nu}\frac{d\theta^{\frac{1}{2}}}{dt}.$$

We will find integral of this magnitude:

$$\int \frac{2t}{\nu}\frac{d\theta^{\frac{1}{2}}}{dt}dt = \frac{2}{\nu}\int t\, d\theta^{\frac{1}{2}} = \frac{2}{\nu}\left(\theta^{\frac{1}{2}}t - \int \theta^{\frac{1}{2}}dt\right),$$

where

$$\int \theta^{\frac{1}{2}}dt = 2\sqrt{c}\int \sqrt{\frac{\nu^2}{16c} - t^{\nu+2}}\,\frac{dt}{t}.$$

Considering

$$t^{\frac{\nu+2}{2}} = u, \quad \frac{\nu}{4\sqrt{c}} = a,$$

we will obtain

$$\int \theta^{\frac{1}{2}}dt = \frac{4\sqrt{c}}{\nu+2}\int \frac{\sqrt{u^2-a^2}}{u}\,du.$$

Applying formula for calculation of last integral [36], we will find

$$\int \theta^{\frac{1}{2}}dt = \frac{4\sqrt{c}}{\nu+2}\left(\sqrt{u^2-a^2} - a\arccos\frac{a}{u} + C\right) =$$
$$= \frac{2t\theta^{\frac{1}{2}}}{\nu+2} - \frac{\nu}{\nu+2}\arccos\frac{a}{u} + C_1.$$

Uniting obtained results, we will come to equalities

$$\int\left(\zeta_2^{(1)} - \zeta_1^{(1)} + h_{22}^{(2)} - h_{11}^{(2)}\right)dt = \frac{2}{\nu}\int t\, d\theta^{\frac{1}{2}} = \frac{2}{\nu+2}\theta^{\frac{1}{2}}t + \frac{2}{\nu+2}\arccos\frac{a}{u} + C_2.$$

where

$$C_2 = -\frac{2t_0}{\nu+2}\theta^{\frac{1}{2}}(t_0) - \frac{2t}{\nu+2}\arccos\frac{a}{u(t_0)}.$$

Let us note that magnitude

$$\zeta_2^{(1)} - \zeta_1^{(1)} + h_{22}^{(2)} - h_{11}^{(2)}$$

is imaginary if $c > 0$ and real if $c < 0$.

Because of found dependence and (given in preceding section) formulas for coefficients $h_{ij}^{(2)}$ $(i, j = 1, 2)$, system (4.51), which weighted canonical components (4.53) satsify, takes the form

$$\left.\begin{aligned} \dot{z}_1 &= \left[\frac{\nu}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-\frac{1}{2}} - \frac{\nu}{4t} + \frac{\dot{\theta}}{4\theta}\right]\times \\ &\times\exp\left(\frac{2t}{\nu+2}\theta^{\frac{1}{2}} + \frac{2t}{\nu+2}\arccos\frac{a}{u} + C_2\right)z_2, \\ \dot{z}_2 &= -\left[\frac{\nu}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-\frac{1}{2}} + \frac{\nu}{4t} - \frac{\dot{\theta}}{4\theta}\right]\times \\ &\times\exp\left(-\frac{2t}{\nu+2}\theta^{\frac{1}{2}} - \frac{2t}{\nu+2}\arccos\frac{a}{u} - C_2\right)z_1. \end{aligned}\right\} \quad (4.54)$$

§ 4. Appraisals of Norm and Phase Coefficients of Particular Solution of a System of Equations Relative to Weighted Canonical Components

System of equations relative to weighted canonical components belongs to class K.

Indeed, after distributing indices with respect to canonical component and (correspondingly) with respect to weighted canonical components and coefficients of corresponding system of equations in such a way that functions $\zeta_1^{(k-1)}(t)$, $\zeta_2^{(k-1)}(t)$, ..., $\zeta_{n-2m}^{(k-1)}(t)$ are real and functions $\zeta_{n-2m+1}^{(k-1)}(t)$, $\zeta_{n-2m+2}^{(k-1)}(t)$, ..., $\zeta_n^{(k-1)}(t)$ – complex and so that for all $i \leq m$ there is executed condition

$$\zeta_{n-2m+i}^{(k-1)} = \bar{\zeta}_{n-m+i}^{(k-1)}.$$

we will obtain in designations of § 8, Chapter III:

a) for coefficients of system (4.51)

$$a_{ij} = h_{ij}^{(k)}\exp\int_{t_0}^{t}\left(\zeta_j^{(k-1)} - \zeta_i^{(k-1)}\right)dt \quad (i, j = 1, \ldots, n);$$

b) for coefficients of system (4.52)

$$a_{ij} = h_{ij}^{(k)}\exp\int_{t_0}^{t}\left(\zeta_j^{(k-1)} + h_{jj}^{(k)} - \zeta_i^{(k-1)} - h_{ii}^{(k)}\right)dt \quad (i, j = 1, \ldots, n;\ i \neq j)$$

and

$$a_{ii}=0 \quad (i=1, \ldots, n).$$

Coefficients a_{ij} in the first case satisfy the condition given in the determination in the shown paragraph, since this condition is satisfied by coefficients $h_{ij}^{(k)}$ (see § 1 of this chapter), and multiplication of these coefficients by shown exponents does not disturb it. Coefficients a_{ii} in the second case satisfy the mentioned condition since they are zero; coefficients a_{ij} $(i \neq j)$ satisfy given condition since this takes place for coefficients $h_{ij}^{(k)}$ and thus, as in the first case, condition is not disturbed during multiplication by corresponding exponents.

We will designate by symbol R norm of solution of system (4.50), by symbols $E_1, \ldots, E_n$ its phase coefficients, and by symbols $F_1, \ldots, F_n$ magnitudes connected with magnitudes $E_1, \ldots, E_n$ by conditions

$$E_i=F_i \quad (i=1, \ldots, n-2m),$$
$$\left.\begin{array}{l} E_j=\frac{1}{\sqrt{2}}(F_j+iF_{j+m}), \\ E_{j+m}=\frac{1}{\sqrt{2}}(F_j-iF_{j+m}) \end{array}\right| \begin{array}{l} (j=n-2m+1,\ n-2m+2, \ldots, n-m) \\ i=\sqrt{-1}. \end{array} \tag{4.55}$$

Since system of equations (4.50) belongs to class K, in the case of real initial values ξ_i to it may be applied method of construction of majorants and minorants of norm of solution and functions $F_1(t), \ldots, F_n(t)$ (which, with such initial conditions, are real), presented in §§ 7, 9, 10, and 11 of the preceding chapter. Since according to definition of weighted canonical components [see formula (4.48)]

$$Z_i(t_0)=z_i(t_0) \quad (i=1, \ldots, n). \tag{4.56}$$

between magnitudes $R(t_0)$ and $r(t_0)$, $F_i(t_0)$ and $f_i(t_0)$ $(i = 1, \ldots, n)$ there occurs connection

$$R(t_0)=r(t_0), \quad F_i(t_0)=f_i(t_0) \quad (i=1, \ldots, n). \tag{4.57}$$

If one were to designate by symbol M_1 minimum characteristic number, and by symbol M_n is maximum characteristic number of matrix, built by coefficients of system (4.50) as in § 7 of the preceding chapter there was constructed matrix

$$\left\| \frac{a_{ij}+\bar{a}_{ji}}{2} \right\|_1^n,$$

then for simplest particular case – majorants and minorants of first approximation of norm R(t) we will obtain formulas

$$\left.\begin{aligned} \mathrm{Maj}_1 R(t) &= r(t_0)\exp\int_{t_0}^{t} M_2\,dt, \\ \mathrm{Min}_1 R(t) &= r(t_0)\exp\int_{t_0}^{t} M_1\,dt. \end{aligned}\right\} \qquad (4.58)$$

In particular, determining approximate solution $\tilde{x}(t)$ by formula (4.21), for case n = 2 according to formulas (4.58) we will find

$$\left.\begin{aligned} &\mathrm{Maj}_1 R(t) = \\ &= r(t_0)\exp\frac{1}{2}\int_{t_0}^{t}\Big|h_{12}^{(k)}\exp\int_{t_0}^{t}(\zeta_2^{(k-1)}+h_{22}^{(k)}-\zeta_1^{(k-1)}-h_{11}^{(k)})\,d\tau+ \\ &+h_{21}^{(k)}\exp\int_{t_0}^{t}(\bar{\zeta}_1^{(k-1)}+\bar{h}_{11}^{(k)}-\bar{\zeta}_2^{(k-1)}-\bar{h}_{22}^{(k)})\,d\tau\Big|\,dt, \\ &\mathrm{Min}_1 R(t) = \\ &= r(t_0)\exp\Big\{-\frac{1}{2}\int_{t_0}^{t}\Big|h_{12}^{(k)}\exp\int_{t_0}^{t}(\zeta_2^{(k-1)}+h_{22}^{(k)}-\zeta_1^{(k-1)}-h_{11}^{(k)})\,d\tau+ \\ &+h_{21}^{(k)}\exp\int_{t_0}^{t}(\bar{\zeta}_1^{(k-1)}+\bar{h}_{11}^{(k)}-\bar{\zeta}_2^{(k-1)}-\bar{h}_{22}^{(k)})\,d\tau\Big|\,dt\Big\}. \end{aligned}\right\} \qquad (4.59)$$

where k is number of considered canonical expansion of solution x(t). If functions $\zeta_1(t)$ and $\zeta_2(t)$ are real, then formulas (4.59) takes the form

$$\left.\begin{aligned} &\mathrm{Maj}_1 R(t) = \\ &= r(t_0)\exp\frac{1}{2}\int_{t_0}^{t}\Big|h_{12}^{(k)}\exp\int_{t_0}^{t}(\zeta_2^{(k-1)}+h_{22}^{(k)}-\zeta_1^{(k-1)}-h_{11}^{(k)})\,d\tau+ \\ &+h_{21}^{(k)}\exp\int_{t_0}^{t}(\zeta_1^{(k-1)}+h_{11}^{(k)}-\zeta_2^{(k-1)}-h_{22}^{(k)})\,d\tau\Big|\,dt, \\ &\mathrm{Min}_1 R(t) = \\ &= r(t_0)\exp\Big[-\frac{1}{2}\int_{t_0}^{t}h_{12}^{(k)}\exp\int_{t_0}^{t}(\zeta_2^{(k-1)}+h_{22}^{(k)}-\zeta_1^{(k-1)}h_{11}^{(k)})\,d\tau+ \\ &+h_{21}^{(k)}\exp\int_{t_0}^{t}(\zeta_1^{(k-1)}+h_{11}^{(k)}-\zeta_2^{(k-1)}-h_{22}^{(k)})\,d\tau\,dt\Big]. \end{aligned}\right\} \qquad (4.60)$$

and if they are complex, then

$$\left.\begin{aligned} \mathrm{Maj}_1 R(t) &= r(t_0)\exp\int_{t_0}^{t}|h_{12}^{(k)}|\,dt, \\ \mathrm{Min}_1 R(t) &= r(t_0)\exp\Big(-\int_{t_0}^{t}|h_{12}^{(k)}|\,dt\Big). \end{aligned}\right\} \qquad (4.61)$$

Example. In the preceding paragraph, as a result of second canonical expansion of solution of equation (4.41), there was obtained a system of equations relative to weighted canonical components (4.54). We will define majorants and minorants of

first approximation of the norm of its solution R(t).

We will be limited by case (4.42), assuming

$$t_0^{\nu+2} > \frac{\nu^2}{16\varepsilon}, \quad \varepsilon > 0. \tag{4.42a}$$

Since during condition (4.42a) functions $\zeta_1^{(1)}(t)$ and $\zeta_2^{(1)}(t)$ are complex, then for determination of unknown magnitudes in arbitrary interval (t_0, t_1) where t_0 satisfies condition (4.42a) and t_1 is any fixed moment of time following moment t_0, it is possible to apply formula (4.61).

We have (see preceding paragraph)

$$\mathrm{Re}\,A_{12}^{(2)} = -\frac{\nu}{4t} + \frac{\dot{\theta}}{4\theta}, \quad \mathrm{Im}\,A_{12}^{(2)} = -\frac{\nu\sqrt{\varepsilon}}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-1/2},$$

$$\sqrt{(\mathrm{Re}\,A_{12})^2 + (\mathrm{Im}\,A_{12})^2} = \frac{\sqrt{\varepsilon}\,\nu(\nu+2)}{16\varepsilon t^{\nu+2} - \nu^2}.$$

Consequently,

$$\left.\begin{aligned} \mathrm{Maj}_1 R(t) &= r(t_0)\exp\int_{t_0}^{t}\sqrt{(\mathrm{Re}\,A_{12})^2 + (\mathrm{Im}\,A_{12})^2}\,dt = \\ &= r(t_0)\exp\left(|\nu|\sqrt{\varepsilon}(\nu+2)\int_{t_0}^{t}\frac{\sqrt{t^{\nu}}}{16\varepsilon t^{\nu+2} - \nu^2}\,dt\right), \\ \mathrm{Min}_1 R(t) &= r(t_0)\exp\left[-\int_{t_0}^{t}\sqrt{(\mathrm{Re}\,A_{12})^2 + (\mathrm{Im}\,A_{12})^2}\,dt\right] = \\ &= r(t_0)\exp\left(-|\nu|\sqrt{\varepsilon}(\nu+2)\int_{t_0}^{t}\frac{\sqrt{t^{\nu}}}{16\varepsilon t^{\nu+2} - \nu^2}\,dt\right). \end{aligned}\right\} \tag{4.62}$$

As a numerical example we will calculate function

$$\mathrm{Maj}_1 R(t) \text{ and } \mathrm{Min}_1 R(t)$$

for case

$$\varepsilon = 1, \quad \nu = 2, \quad t_0 = 1, \quad t_1 = 8, \quad x(t_0) = 1, \quad \dot{x}(t_0) = 0.$$

By formulas given in § 2, functions $\zeta_1^{(1)}(t)$ and $\zeta_2^{(1)}(t)$ we will obtain in the form

$$\zeta_{1,2}^{(1)}(t) = -\frac{1}{2t} \mp \frac{1}{2}\sqrt{\frac{1}{t^2} - 4t^2}.$$

On the basis of system (2.44), canonical components $z_1(t)$ and $z_2(t)$ are connected with functions $x(t)$ and $\dot{x}(t)$ by dependence

$$\left.\begin{aligned} z_1(t) &= \frac{\dot{x} - \zeta_2^{(1)}x}{\zeta_1^{(1)} - \zeta_2^{(1)}}, \\ z_2(t) &= \frac{\dot{x} - \zeta_1^{(1)}x}{\zeta_2^{(1)} - \zeta_1^{(1)}}. \end{aligned}\right\}$$

Using expression found above for functions $\zeta_1(t)$ and $\zeta_2(t)$ and given initial values of magnitudes x and $\dot{x}$, we will obtain

$$\left.\begin{aligned} z_1(1) &= \frac{1}{2}\left(1 - \frac{i}{\sqrt{3}}\right), \\ z_2(1) &= \frac{1}{2}\left(1 + \frac{i}{\sqrt{3}}\right). \end{aligned}\right\}$$

Hence

$$r(1) = \sqrt{2z_1(1)\bar{z}_1(1)} = \sqrt{\frac{2}{3}}.$$

Further at $\nu = 2$, $c = 1$, we will find

$$\nu\sqrt{c}(\nu+2) = 8.$$

$$\frac{\sqrt{t^{\nu}}}{16ct^{\nu+2} - \nu^2} = \frac{t}{16t^4 - 4}$$

Now, when all magnitudes in the right sides of equalities (4.62) are determined it is possible to calculate functions $\mathrm{Maj}_1 R(t)$ and $\mathrm{Min}_1 R(t)$. For considered values ν and c after integration formulas (4.62) take the form

$$\mathrm{Maj}_1 R(t) = r(t_0)\left[\frac{(1-2t^2)(1+2t_0^2)}{(1+2t^2)(1-2t_0^2)}\right]^{\frac{1}{4}},$$

$$\mathrm{Min}_1 R(t) = r(t_0)\left[\frac{(1+2t^2)(1-2t_0^2)}{(1-2t^2)(1+2t_0^2)}\right]^{\frac{1}{4}}.$$

Results of calculation are presented in Table 4.

Table 4.

t	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4	2,6	2,8
$\mathrm{Maj}_1 R$	0,816	0,890	0,948	0,991	1,010	1,029	1,037	1,050	1,058	1,065
$\mathrm{Min}_1 R$	0,816	0,750	0,702	0,672	0,660	0,648	0,642	0,635	0,630	0,625
t	3,0	3,2	3,4	3,6	3,8	4,0	4,2	4,4	4,6	4,8
$\mathrm{Maj}_1 R$	1,070	1,072	1,078	1,081	1,083	1,084	1,085	1,088	1,090	1,090
$\mathrm{Min}_1 R$	0,623	0,621	0,618	0,615	0,615	0,614	0,614	0,613	0,611	0,611
t	5,0	5,2	5,4	5,6	5,8	6,0	6,2	6,4	6,6	6,8
$\mathrm{Maj}_1 R$	1,090	1,090	1,091	1,092	1,092	1,094	1,094	1,096	1,098	1,098
$\mathrm{Min}_1 R$	0,611	0,611	0,610	0,610	0,610	0,610	0,610	0,608	0,607	0,607

(Table 4 Cont'd)

t	7,0	7,2	7,4	7,6	7,8	8,0
$\mathrm{Maj}_1 R$	1,099	1,100	1,100	1,100	1,100	1,100
$\mathrm{Min}_1 R$	0,606	0,606	0,606	0,606	0,606	0,606

§ 5. Definitized Majorants and Minorants of Real Particular Solution of an Equation of Oscillations

If appraisals for norm and phase coefficients of solution of a system of equations relative to weighted canonical components are found, then there can be built a corresponding pair of majorants and minorants of real particular solution of equation of oscillations (0.1).

Because of equations (4.48), solution of equation (0.1) is connected with weighted canonical components $Z_1(t)$, ..., $Z_n(t)$ by dependence

$$x(t)=\sum_{i=1}^{n}\frac{\tilde{x}_i(t)}{C_i}Z_i(t). \tag{4.63}$$

If approximate solutions $\tilde{x}_i(t)$ ($i = 1, ..., n$) are determined by formulas (4.23) then this dependence has the form

$$x(t)=\sum_{i=1}^{n}Z_i(t)\exp\int_{t_0}^{t}\zeta_i^{(k-1)}(t)\,dt; \tag{4.64}$$

during determination of approximate solutions $\tilde{x}_i(t)$ by formulas (4.24) this dependence takes the form

$$x(t)=\sum_{i=1}^{n}Z_i\exp\int_{t_0}^{t}(\zeta_i^{(k-1)}+h_i^{(k)})\,dt. \tag{4.65}$$

Let us assume that there are known (corresponding to the considered, real, particular solution of equation (0.1)) majorants and minorants of l-th approximation of function $F_1(t)$, ..., $F_n(t)$ and majorant and minorant of p-th approximation of function $R(t)$. Let us consider two methods of determination, by these data, of majorant and minorant of mentioned solution.

1. Equation (4.63) it is possible to present in the form

$$x(t)=R\sum_{i=1}^{n}\frac{\tilde{x}_i}{C_i}E_i. \tag{4.66}$$

Let us note that magnitudes $\frac{\tilde{x}_i(t)}{\tilde{C}_i}$ and $E_i(t)$ ($i = 1, \ldots, n$) are real if $i \leq n - 2m$, and are complex if $i > n - 2m$; in the latter case magnitudes $\tilde{x}_i/C_i$ and $\tilde{x}_{i+m}/C_{i+m}$, E_i and E_{i+m} are conjugate. After crossing from magnitudes E_i ($i = 1, \ldots, n$) to magnitudes F_i by formulas (4.55) and presenting magnitudes $\tilde{x}_i C_i$ during $n - 2m < i \leq n - m$ in the form

$$\frac{\tilde{x}_i}{C_i} = \operatorname{Re}\frac{\tilde{x}_i}{C_i} + \sqrt{-1}\operatorname{Im}\frac{\tilde{x}_i}{C_i} \quad (i = n-2m+1, \ldots, n-m). \tag{4.67}$$

because of equality (4.66) we will obtain

$$x(t) = R\left(\sum_{i=1}^{n-2m}\frac{\tilde{x}_i F_i}{C_i} + \sqrt{2}\sum_{i=n-2m+1}^{n-m}\operatorname{Re}\frac{\tilde{x}_i F_i}{C_i} + \sqrt{2}\sum_{i=n-2m+1}^{n}\operatorname{Im}\frac{\tilde{x}_i F_i}{C_i}\right). \tag{4.68}$$

Considering inequality

$$\operatorname{Min}_i F_i(t) < F_i(t) < \operatorname{Maj}_i F_i(t) \quad (i = 1, \ldots, n),$$

$$\operatorname{Min}_p R(t) < R(t) < \operatorname{Maj}_p R(t),$$

equality

$$\sum_{i=1}^{n} F_i^2 = 1$$

and equality (4.68), one can determine majorants and minorants of function x(t):

$$\left.\begin{aligned}
\operatorname{Maj} x(t) &= \max\left(\operatorname{Min}_p R \max_{\substack{\|F\|=1 \\ \operatorname{Min}_i F < F < \operatorname{Maj}_i F}} \sum_{i=1}^{n}\tilde{X}_i F_i,\ \operatorname{Maj}_p R \max_{\substack{\|F\|=1, \\ \operatorname{Min}_i F < F < \operatorname{Maj}_i F}} \sum_{i=1}^{n}\tilde{X}_i F_i\right). \\
\operatorname{Min} x(t) &= \min\left(\operatorname{Min}_p R \min_{\substack{\|F\|=1, \\ \operatorname{Min}_i F < F < \operatorname{Maj}_i F}} \sum_{i=1}^{n}\tilde{X}_i F_i,\ \operatorname{Maj}_p R \min_{\substack{\|F\|=1 \\ \operatorname{Min}_i F < F < \operatorname{Maj}_i F}} \sum_{i=1}^{n}\tilde{X}_i F_i\right).
\end{aligned}\right\} \tag{4.69}$$

where

$$\|F\| = \sqrt{\sum_{i=1}^{n} F_i^2}$$

$\mathrm{Min}_l F \leq F \leq \mathrm{Maj}_l F$ – in abbreviated recording system of inequalities

$$\mathrm{Min}_i F_i < F_i < \mathrm{Maj}_i F_i$$
$$(i = 1, \ldots, n);$$

$$\tilde{x}_i(t) = \begin{cases} \dfrac{\tilde{x}_i}{C_i} \text{ — when } i \leqslant n-2m, \\ = \sqrt{2}\,\mathrm{Re}\,\dfrac{\tilde{x}_i}{C_i} \text{ — when } n-2m < i \leqslant n-m, \\ -\sqrt{2}\,\mathrm{Im}\,\dfrac{\tilde{x}_i}{C_i} \text{ — when } n-m < i \leqslant n. \end{cases}$$

2. From equations (4.8) and (4.48) follows

$$\dot{x} = \sum_{i,j=1}^{n} \frac{(\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)})\,x_i}{C_i}\,Z_i = -R \sum_{i,j=1}^{n} \frac{(\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)})\,x_i}{C_i}\,E_i. \tag{4.70}$$

Crossing from magnitude E_i ($i = 1, \ldots, n$) to magnitudes F_i by formulas (4.55), presenting magnitude $\frac{\tilde{x}_i(t)}{C_i}$ in the form of sums (4.67) and in the form of analogous sums of magnitude

$$\sum_{j=1}^{n} (\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)}) \qquad (i = 1, \ldots, n),$$

because of equality (4.70) we will obtain

$$\dot{x} = R \sum_{i=1}^{n} Q_i F_i, \tag{4.71}$$

where

$$Q_i = \begin{cases} -\dfrac{\tilde{x}_i}{C_i} \sum\limits_{j=1}^{n} (\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)}) \quad \text{— when } i \leqslant n-2m, \\ -\sqrt{2}\left[\mathrm{Re}\,\dfrac{\tilde{x}_i}{C_i}\,\mathrm{Re} \sum\limits_{j=1}^{n} (\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)}) - \right. \\ \left. -\mathrm{Im}\,\dfrac{\tilde{x}_i}{C_i}\,\mathrm{Im} \sum\limits_{j=1}^{n} (\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)})\right] \text{ — when } n-2m < i \leqslant n-m, \\ -\sqrt{2}\left[\mathrm{Re}\,\dfrac{\tilde{x}_i}{C_i}\,\mathrm{Im} \sum\limits_{j=1}^{n} (\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)}) + \right. \\ \left. +\mathrm{Im}\,\dfrac{\tilde{x}_i}{C_i}\,\mathrm{Re} \sum\limits_{j=1}^{n} (\zeta_i^{(k-1)}\delta_{ij} + h_{ji}^{(k)})\right] \text{ — when } n-m < i \leqslant n. \end{cases}$$

Designating magnitude

$$\min \sum_{i=1}^{n} Q_i' F_i,$$

found during conditions

$$\left.\begin{array}{l}\sum_{i=1}^{n} F_i^2 = 1, \\ \mathrm{Min}_l F_i \leqslant F_i \leqslant \mathrm{Maj}_l F_i,\end{array}\right\} \qquad (4.72)$$

by symbol χ_l' and magnitude

$$\max \sum_{i=1}^{n} Q_i' F_i,$$

(4.71) we will obtain

$$R(t)\chi_l'(t) \leqslant \dot{x}(t) \leqslant \xi_l'(t) R(t). \qquad (4.73)$$

Estimating function R(t) by minorant and majorant of the p-th approximation, we will obtain the following formulas for minorant and majorant of function x(t):

$$\left.\begin{array}{l}\mathrm{Min}\, x(t) = x(t_0) + \int_{t_0}^{t} \min[\chi_l'(\tau)\, \mathrm{Min}_p R(\tau), \\ \chi_l'(\tau)\, \mathrm{Maj}_p R(\tau)]\, d\tau; \\ \mathrm{Maj}\, x(t) = x(t_0) + \int_{t_0}^{t} \max[\xi_l'(\tau)\, \mathrm{Min}_p R(\tau), \\ \xi_l'(\tau)\, \mathrm{Maj}_p R(\tau)]\, d\tau.\end{array}\right\} \qquad (4.74)$$

Minorant and majorant appraisals (4.69) and (4.74) possess the following interesting property: they coincide with solution x(t) if approximate particular solutions $\tilde{x}_j(t)$ (j = 1, ..., n) are exact particular solutions of equation (0.1).

Really, if shown condition is executed, then system of equations relative to weighted canonical components has the form (4.50). With this,

$$\mathrm{Min}_p R = R = \mathrm{Maj}_p R,$$
$$\mathrm{Min}_l F_i = F_i = \mathrm{Maj}_l F_i \quad (i = 1, \ldots, n)$$

during any l, p ≥ 1. The validity of the expressed affirmation in the case of formulas (4.69) is evident. In the case of formulas (4.74) it is necessary, preliminarily, to clarify relationship between functions $\chi_l'(t)$ and $\xi_l'(t)$. Due to the last equalities $\chi_l'(t) = \xi_l'(t)$. Considering this, by formulas (4.74) we will find

$$\mathrm{Min}\, x(t) = \mathrm{Maj}\, x(t).$$

Hence is evident the validity of the expressed affirmation in this case.

Because of the considered property of (fixed in given paragraph) majorants and minorants during equal conditions of determination (for the same numbers l and m),

they, as a rule, are more exact than majorants and minorants of particular solutions delivered by appraisals fixed in § 1.

Equalities (4.69) and (4.74) determine two new classes of majorants and minorants of particular solution of an equation of oscillations. Considering majorants and minorants of functions $F_i(t)$ of zero ($MajF_1 = 1$, $MinF_1 = -1$), first, second, ..., l-th approximation and majorants and minorants of function $R(t)$ of first, second, ..., m-th approximation, considering $m = l$ or $m = l + 1$ and increasing l, we will obtain, on the basis of each of the shown systems of equalities sequences of majorants and minorants. These sequences converge to solution at $l \to \infty$.

Example: We will construct majorants and minorants of particular solution of equation

$$\ddot{x} + t^2 x = 0 \tag{4.75}$$

in interval (1,8) during initial conditions

$$x(1) = 1, \quad \dot{x}(1) = 0.$$

We will be limited by majorants and minorants of first approximation, using results, introduced in preceding paragraph, of calculations of functions $Min_1 R(t)$ and $Maj_1 R(t)$. Relative to functions $F_i(t)$ ($i = 1, \ldots, n$) we will assume that they can take arbitrary values in section [-1, 1], in other words, we will consider $l = 0$. Approximate solutions $\tilde{x}_{1,2}(t)$ we will determine by formulas

$$\tilde{x}_{1,2}(t) = \tilde{x}_{1,2}(t_0)\exp\int_{t_0}^{t}\left(c_{1,2}^{(1)} + h_{11,22}^{(1)}\right)dt,$$

which, because of calculations introduced in §§ 2 and 3, take the form[1]

$$\tilde{x}_{1,2}(t) = \frac{1}{2}\left(1 \pm \frac{i}{\sqrt{3}}\right)\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4}\exp\left[\mp\frac{i}{4}\left(\sqrt{4t^4-1} + \arccos\frac{1}{2t^2} - \sqrt{3} - \frac{\pi}{3}\right)\right].$$

where

$$\theta(t) = \frac{1}{t^2} - 4t^2.$$

For calculation of majorants and minorants of the considered solution by the first method, we will define, preliminarily, function $\tilde{X}_1(t)$ and $\tilde{X}_2(t)$. According

[1]This formula follows from formulas obtained in § 3 for integral

$$\int\left(c_1^{(1)} - c_2^{(1)} + h_{22}^{(1)} - h_{11}^{(1)}\right)dt,$$

if one were to assume $c = 1$, $\nu = 2$ and to consider that

$$c_{1,2}^{(1)} + h_{11,22}^{(1)} = \pm\frac{1}{2}\left(c_1^{(1)} - c_2^{(1)} - h_{22}^{(1)} - h_{11}^{(1)}\right) - \frac{1}{4}\frac{d\ln\theta}{dt}$$

(see § 2).

to the formula for these magnitudes, given in explanations to formulas (4.69), and obtained expression for solutions $\widetilde{X}_1(t)$ and $\widetilde{X}_2(t)$, we will find

$$\widetilde{x}_1(t) = \sqrt{2}\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4} \cos\frac{1}{4}\left(\sqrt{4t^4-1} + \arc\cos\frac{1}{2t^2} - \sqrt{3} - \frac{\pi}{3}\right),$$

$$\widetilde{x}_2(t) = \sqrt{2}\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4} \sin\frac{1}{4}\left(\sqrt{4t^4-1} + \arc\cos\frac{1}{2t^2} - \sqrt{3} - \frac{\pi}{3}\right).$$

Formulas (4.69) for $m = 1$ and $l = 0$ take the form

$$\mathrm{Maj}\, x(t) = \|\widetilde{x}\|\, \mathrm{Maj}_1 R,$$
$$\mathrm{Min}\, x(t) = -\|\widetilde{x}\|\, \mathrm{Maj}_1 R,$$

where

$$\|\widetilde{x}\| = \sqrt{\sum_{i=1}^{n} \widetilde{x}_i^2}.$$

Applying these formulas to the considered example and using the above formula for functions $\widetilde{X}_1(t)$ and $\widetilde{X}_2(t)$, we will obtain

$$\left.\begin{aligned} \mathrm{Maj}\, x(t) &= \sqrt{2}\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4} \mathrm{Maj}_1 R, \\ \mathrm{Min}\, x(t) &= -\sqrt{2}\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4} \mathrm{Maj}_1 R. \end{aligned}\right\} \tag{4.76}$$

Results of calculation of function $\mathrm{Maj}_1 R(t)$ are given in preceding paragraph. Using them, by formulas (4.76) we will find majorant and minorant of function $x(t)$. Results of calculation are given in Table 5.

Table 5.

t	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4	2,6	2,8
Min x	−1,14	−1,11	−1,08	−1,04	−1,00	−0,96	−0,93	−0,90	−0,87	−0,84
Maj x	1,14	1,11	1,08	1,04	1,00	0,96	0,93	0,90	0,87	0,84
t	3,0	3,2	3,4	3,6	3,8	4,0	4,2	4,4	4,6	4,8
Min x	−0,82	−0,79	−0,77	−0,75	−0,73	−0,72	−0,70	−0,68	−0,67	−0,66
Maj x	0,82	0,79	0,77	0,75	0,73	0,72	0,70	0,68	0,67	0,66
t	5,0	5,2	5,4	5,6	5,8	6,0	6,2	6,4	6,6	6,8
Min x	−0,64	−0,64	−0,62	−0,61	−0,60	−0,59	−0,58	−0,57	−0,56	−0,56
Maj x	0,64	0,64	0,62	0,61	0,60	0,59	0,58	0,57	0,56	0,56

(Table 5 Cont'd)

t	7,0	7,2	7,4	7,6	7,8	8,0
Min x	−0,55	−0,54	−0,53	−0,53	−0,52	−0,51
Maj x	0,55	0,54	0,53	0,53	0,52	0,51

For determination of majorants and minorants of solution by the second method it is necessary, preliminarily, to calculate functions $\chi_0'(t)$ and $\xi_0'(t)$. Since in the determination of these functions participate magnitudes $Q_1'(t)$ and $Q_2'(t)$, we will define, first of all, the last ones. By the formulas given in explanations to formulas (4.71), for the considered example we will obtain

$$
\begin{aligned}
Q_1' = -\frac{\sqrt{2}}{2t}\Bigg(&\cos\frac{\sqrt{4t^4-1}+\arccos\frac{1}{2t^2}-\sqrt{3}-\frac{\pi}{3}}{4} + \\
&+\frac{3+4t^4}{\sqrt{4t^4-1}}\sin\frac{\sqrt{4t^4-1}+\arccos\frac{1}{2t^2}-\sqrt{3}-\frac{\pi}{3}}{4}\Bigg)\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4}, \\
Q_2' = \frac{\sqrt{2}}{2t}\Bigg(&\frac{3+4t^4}{\sqrt{4t^4-1}}\cos\frac{\sqrt{4t^4-1}+\arccos\frac{1}{2t^2}-\sqrt{3}-\frac{\pi}{3}}{4} - \\
&-\sin\frac{\sqrt{4t^4-1}+\arccos\frac{1}{2t^2}-\sqrt{3}-\frac{\pi}{3}}{4}\Bigg)\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4}.
\end{aligned}
\qquad (4.77)
$$

Conditions (4.72) in this case take the form

$$F_1^2+F_2^2=1.$$

Under this condition minimum of magnitude $Q_1'F_1 + Q_2'F_2$ is equal to magnitude

$$-\sqrt{(Q_1')^2+(Q_2')^2}.$$

and maximum is magnitude

$$\sqrt{(Q_1')^2+(Q_2')^2}.$$

Consequently

$$\chi_0' = -\sqrt{(Q_1')^2+(Q_2')^2}.$$

$$\zeta_0' = \sqrt{(Q_1')^2+(Q_2')^2}.$$

Using formula (4.77), we will find

$$\zeta_0' = -\chi_0' = \frac{2}{t}\sqrt{\frac{1+7t^4+2t^8}{4t^4-1}}\left[\frac{\theta(1)}{\theta(t)}\right]^{1/4}. \qquad (4.78)$$

Calculating functions $\zeta_0'(t)$ and $\chi_0'(t)$, by formulas (4.74), we will calculate functions Min x(t) and Maj x(t) (assuming $l = 0$, $m = 1$). Results of calculation for section [1.5] are given in Table 6.

Table 6.

t	1,0	1,2	1,4	1,6	1,8	2,0	2,2	2,4	2,6	2,8	3,0
Min x	1,00	0,47	0,02	−0,40	−0,82	−1,24	−1,68	−2,12	−2,58	−3,05	−3,54
Maj x	1,00	1,53	1,98	2,40	2,82	3,24	3,68	4,12	4,58	5,05	5,54

t	3,2	3,4	3,6	3,8	4,0	4,2	4,4	4,6	4,8	5,0
Min x	−4,05	−4,58	−5,12	−5,66	−6,23	−6,82	−7,42	−8,01	−8,66	−9,23
Maj x	6,05	5,58	6,12	6,66	7,23	7,82	8,42	9,04	9,66	10,23

Graphs of minorants and majorants of considered solution of equation (4.75), calculated by first and second methods, are depicted on Fig. 12. In the same place are graphs of solution found by numerical integration.

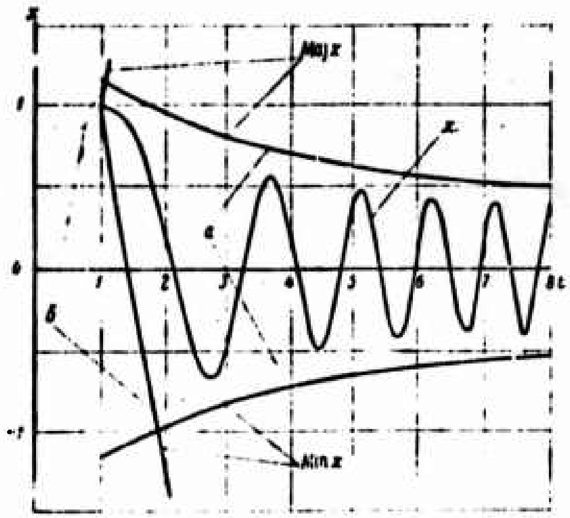

Fig. 12. Majorants and minorants of particular solution of equation $x + t^2x = 0$.

CHAPTER V

STABILITY OF OSCILLATIONS

§ 1. Ideas of Stability and Instability of Oscillations

A. M. Lyapunov [6] investigated certain properties of dynamic systems, the motion of which obeys system of differential equations (1.5)

$$\dot{x}_i = X_i(x_1, \ldots, x_n, t) \qquad (i = 1, \ldots, n). \tag{5.1}$$

on the assumption that variables $x_1, \ldots, x_n$, determining state of system, remain real for any real states of system, and functions $X_1(x_1, \ldots, x_n, t), \ldots, X_n(x_1, \ldots, x_n, t)$ are determined during all values $x_1, \ldots, x_n$ belonging to certain region G, during all values of t belonging to half-open interval [0, ∞], and are continuous in shown region of variation of arguments with respect to variables $x_1, \ldots, x_n$.

Interesting A. M. Lyapunov, the properties of considered systems were included in the stability of their undisturbed motion.

Under undisturbed motion of system, A. M. Lyapunov understood one of its possible motions, represented by selected particular solution of system of equations (5.1)

$$x_1 = f_1(t),\ x_2 = f_2(t), \ldots, x_n = f_n(t)$$

and definite for any $t \geq t_0$, where t_0 is given moment of time.

Ideas of stability of undisturbed motion A. M. Lyapunov connected with properties of disturbed motion, understanding by this any other motion obeying the same system of differential equations considered from the same moment of time t_0 but differing from undisturbed.

Obviously, with this definition of the ideas of undisturbed and disturbed motions, they can only be distinguished because of initial values of variables $x_1, x_2, \ldots, x_n$.

We will designate initial values of shown magnitudes in undisturbed motion by symbols

$$f_1^{(0)}, f_2^{(0)}, \ldots, f_n^{(0)},$$

initial values of the same magnitudes in disturbed motion by symbols

$$x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}.$$

Differences

$$\varepsilon_i = x_i^{(0)} - f_i^{(0)} \qquad (i = 1, 2, \ldots, n)$$

are called initial disturbances.

The idea of stability of undisturbed motion A. M. Lyapunov determined with respect to magnitudes

$$Q_1, Q_2, \ldots, Q_p,$$

being given functions of variables x_1, x_2, ..., x_n.

Undisturbed motion is stable with respect to magnitudes Q_1, Q_2, ..., Q_p if, assigning arbitrary positive numbers L_1, L_2, ..., L_p, we can determine during any L_i ($i = 1, 2, \ldots, p$), however small they may be, such positive numbers

$$E_1, E_2, \ldots, E_n,$$

so that during any real values of magnitudes

$$\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n,$$

satisfying conditions

$$|\varepsilon_i| \leqslant E_i \qquad (i = 1, 2, \ldots, n),$$

and at any t exceeding t_0, there is executed inequality

$$|Q_1 - Q_1^{(0)}| < L_1,\ |Q_2 - Q_2^{(0)}| < L_2, \ldots, |Q_p - Q_p^{(0)}| < L_p,$$

where $Q_1^{(0)}$, $Q_2^{(0)}$, ..., $Q_p^{(0)}$ are values of magnitudes Q_1, Q_2, ..., Q_p in undisturbed motion; if shown condition is not executed, undisturbed motion is unstable with respect to magnitudes Q_1, Q_2, ..., Q_p.

As follows from the definition, the idea of stability of undisturbed motion with respect to magnitudes Q_1, Q_2, ..., Q_p is connected with the properties of these magnitudes as functions of t, in which they are transformed during presentation of magnitudes x_1, x_2, ..., x_n by functions of t, satisfying system of differential equations (5.1) and having initial values sufficiently close to initial values taken by these variables in undisturbed motion. With this, motion may be stable with respect to one system of magnitudes Q_1, Q_2, ..., Q_p but unstable with respect to another.

In this work we will study stability of undisturbed motion of linear dynamics systems whose law of motion is expressed by equations (1.10)

$$\dot{x}_i = a_{i1}(t)x_1 + \ldots + a_{in}(t)x_n + Y_i(t) \qquad (i=1, \ldots, n). \tag{5.2}$$

As undisturbed motion we will consider any state of rest of system

$$\dot{x}_i = 0 \quad (i=1, \ldots, n).$$

We will study stability of this state with respect to magnitude x_1 or, which is equivalent, with respect to its deviation from value in state of rest Δx_1. Equation of free oscillations (0.1) we will consider recorded relative to last magnitude, taking, thus, designation $\Delta x_1 = x$.

Obviously, any solution of system (5.2), presenting state of rest of investigated dynamic system, corresponds to zero solution of equation (0.1):

$$x \equiv 0.$$

We will assume in above-mentioned definition of the idea of stability, given by A. M. Lyapunov, $p = 1$ and $Q = \Delta x_1 = x$, and we will assume that between systems of initial values of variables $x_1, \ldots, x_n$, determining particular solutions of system (5.1), and systems of initial values of magnitudes $x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}$, there exists a mutual-simple dependence, where system of limited values of some magnitudes corresponds to the system of limited values of other magnitudes. Then the definition of the idea of stability of undisturbed motion with respect to magnitude x it is possible to give the following form.

Definition. Undisturbed motion, presented by zero solution of equation (0.1), is stable with respect to magnitude x if, assigning arbitrarily positive number L, we can determine during any L, however small it may be, such positive numbers E_0, $E_1, \ldots, E_{n-1}$, so that during any initial values of magnitudes $x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}$, satisfying conditions

$$|x(t_0)| \leqslant E_0, \quad \left|\frac{dx}{dt}\right|_{t=t_0} \leqslant E_1, \ldots, \left|\frac{d^{n-1}x}{dt^{n-1}}\right|_{t=t_0} < E_{n-1}, \tag{5.3}$$

and during any t exceeding t_0, there is executed inequality

$$|x| < L.$$

If shown condition is not executed, undisturbed motion is unstable with respect to magnitude x.

In this definition, system of numbers $E_0, E_1, \ldots, E_{n-1}$ it is possible to replace by one number E, and inequality (5.3) by inequalities

$$|x(t_0)| \leqslant E, \quad \left|\frac{dx}{dt}\right|_{t=t_0} \leqslant E, \ldots, \left|\frac{d^{n-1}x}{dt^{n-1}}\right|_{t=t_0} \leqslant E. \tag{5.4}$$

This definition does not connect the property of stability of undisturbed motion with the requirement of obligatory approach of disturbed motions to undisturbed during $t \to \infty$.

Let us assume that undisturbed motion is stable with respect to magnitude x and numbers E_0, E_1, ..., E_{n-1} in inequalities (5.3) or number E in inequalities (5.4) are determined. Then two cases are possible:

a) for all solutions x(t) emanating from region of initial values of variables x and its derivatives, determined by inequalities (5.3) or (5.4), there is executed condition

$$\lim_{t\to\infty} |x| = 0;$$

b) there exist solutions x(t) emanating from shown region, for which this condition is not executed.

If first case takes place, then, according to determination of Lyapunov, [6] undisturbed motion we will call asymptotically stable with respect to magnitude x.

With respect to property of solutions of linear differential equations, the difference of solutions of equation (0.1) also is its solution. Therefore, considering arbitrary, non-zero, particular solution of equation (0.1) $x^{(0)}(t)$ and carrying out in inequalities (5.3) and (5.4) substitutions

$$x = x' - x^{(0)}, \quad \frac{dx}{dt} = \frac{dx'}{dt} - \frac{dx^{(0)}}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}} = \frac{d^{n-1}x'}{dt^{n-1}} - \frac{d^{n-1}x^{(0)}}{dt^{n-1}}$$

where $x'(t)$ is a certain new solution of equation (0.1), because of this property and given definition we will find that if state of rest is stable, asymptotically stable or unstable with respect to magnitude x, then, correspondingly, undisturbed motion, presented by arbitrary particular solution of equation of free oscillations, is stable, asymptotically stable, or unstable with respect to this magnitude.

The shown relationship between properties of the state of rest and arbitrary undisturbed motion allows us not to connect the idea of stability, asymptotic stability, and instability of undisturbed motion with respect to magnitude x with any specific undisturbed motion, inserting in the idea of undisturbed motion the meaning of an arbitrary process of free oscillations of a studied system which may be, in particular, also a state of rest. Introducing, logically, the terminological simplifications following from this and omitting the phrase "with respect to magnitude x," stability, and instability of undisturbed motion with respect to magnitude x we will subsequently call, correspondingly, stability, asymptotic stability, and instability of oscillations.

Formulated above, the definitions of ideas of stability, instability, and asymptotic stability of a state of rest it is possible to simplify, using the following properties of solutions of linear, differential equation (0.1): for arbitrary positive L it is possible to select such positive E that from inequalities (5.4) will follow inequality

$$|x|<L$$

in that case and only in that case when all particular solutions of equation (0.1) emanating from arbitrary final n-dimensional region of beginning values of magnitudes $x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}$, are limited. Leaning on this property of solutions of equation (0.1) and replacing (in the mentioned-above definitions) state of rest by arbitrary process of oscillations, we will come to the following definition.

Definition. Oscillations presented by an arbitrary particular solution of equation (0.1) are stable with respect to magnitude x if all particular solutions of this equation, emanating from limited n-dimensional region of initial values of $x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}$, are limited, and they are unstable if among such particular solutions there exists an unlimited solution; oscillations are asymptotically stable if for all shown solutions

$$\lim_{t\to\infty}|x(t)|=0.$$

Subsequently, operating by equation of free oscillations, we will everywhere assume that its coefficients are determined in considered half-open interval which may be either interval $(0, \infty)$ or its unlimited, on the right, part, and they possess properties which were mentioned in § 4, Chapter I, are continuous in this interval and are differentiable in corresponding open interval. All the above-mentioned definitions of this paragraph we will consider valid for free oscillations whose coefficients of equations possess the properties shown here.

§ 2. Sufficient Conditions of Stability, Based on Properties of Canonical Components

Let us assume that coefficients of equation (0.1) are such that in certain interval (t_0, ∞) it is possible to apply some canonical expansion. Then the study of stability of oscillations it is possible to connect with the study of functions S(t) (see § 7, Chapter III), built on coefficients of system of equations relative to canonical components of solution in the considered expansion.

Because of determination of function S(t) and inequalities (4.7) arbitrary particular solution of equation (0.1) satisfies inequality

$$|x(t)| \leqslant \sqrt{n}\, S(t). \tag{5.5}$$

Due to this inequality for stability of oscillations it is sufficient that during $r_0 \neq 0$ function S(t) be limited in interval (t_0, ∞). In accordance with equations (3.47), we will find that for stability of oscillations it is sufficient that the following inequality be executed

$$\overline{\lim_{t\to\infty}} \int_{t_0}^{t} \mu_n\, dt < \infty, \tag{5.6}$$

where μ_n is the biggest characteristic number of matrix

$$\left\| \frac{a_{ij} + \overline{a_{ji}}}{2} \right\|_1^n,$$

coefficients a_{ij} (i, j = 1, ..., n) of which are coefficients of a system of equations relative to canonical components, obtained as a result of considered canonical expansion of solution of equation of oscillations.

For asymptotic stability of oscillations it is sufficient that function S(t) during $r_0 \neq 0$ approach zero at $t \to \infty$. Consequently, for asymptotic stability of oscillations, fulfillment of the following inequality is sufficient

$$\lim_{t\to\infty} \int_{t_0}^{t} \mu_n\, dt = -\infty. \tag{5.7}$$

Conditions (5.6) and (5.7) of stability and asymptotic stability are executed if in certain interval (T, ∞) there are executed correspondingly conditions

$$\mu_n \leqslant 0. \tag{5.8}$$

$$\mu_n < -G < 0, \tag{5.9}$$

where

$$G = \text{const}.$$

Consequently, conditions (5.8) and (5.9) also are sufficient conditions of stability and asymptotic stability (correspondingly). They do not allways allow us to reveal stability or asymptotic stability in those cases when conditions (5.6) or (5.7) lead to this goal; however, frequently they are more convenient in application.

Let us consider these conditions more specifically.

Condition (5.8) is executed if and only if Hermitian form

$$\frac{1}{2} \sum (a_{ij} + \bar{a}_{ji}) e_i \bar{e}_j$$

is nonpositive (see § 6, Chapter III). Condition (5.9) is executed if and only if form

$$\sum\left(\frac{a_{ij}+\bar a_{ji}}{2}+\delta_{ij}G\right)c_i\bar c_j,$$

where $\delta_{ij}=1$ during $i=j$ and $\delta_{ij}=0$ during $i\neq j$) is negatively determined (see in the same place). In accordance with criterion of nonpositivity of Hermitian form, for execution of condition (5.8) it is necessary and sufficient that matrix

$$\left\|\frac{a_{ij}+\bar a_{ji}}{2}\right\|_1^n$$

be such that all its principal minors of even order are nonnegative, and all principal minors of odd order are nonpositive. Because of criterion of negative determinancy of Hermitian form, for execution of condition (5.9) it is necessary and sufficient that there be executed inequalities

$$a_{11}+\bar a_{11}+2G<0,\quad \begin{vmatrix} a_{11}+\bar a_{11}+2G & a_{12}+\bar a_{21}\\ a_{21}+\bar a_{12} & a_{22}+\bar a_{22}+2G\end{vmatrix}>0$$

$$\begin{vmatrix} a_{11}+\bar a_{11}+2G & a_{12}+\bar a_{21} & a_{13}+\bar a_{31}\\ a_{21}+\bar a_{12} & a_{22}+\bar a_{22}+2G & a_{23}+\bar a_{32}\\ a_{31}+\bar a_{13} & a_{32}+\bar a_{23} & a_{33}+\bar a_{33}+2G\end{vmatrix}<0,\ \ldots,(-1)^n\times$$

$$\times\begin{vmatrix} a_{11}+\bar a_{11}+2G & \ldots & a_{1n}+\bar a_{n1}\\ \ldots & \ldots & \ldots\\ a_{n1}+\bar a_{1n} & \ldots & a_{nn}+\bar a_{nn}+2G\end{vmatrix}>0.$$

In particular, in the case of an equation of oscillations of the second order, condition (5.8) is reduced to inequalities

$$a_{11}+\bar a_{11}\leqslant 0,\quad a_{22}+\bar a_{22}\leqslant 0,\quad (a_{11}+\bar a_{11})(a_{22}+\bar a_{22})-|a_{12}+\bar a_{21}|^2\geqslant 0,$$

equivalent to inequalities $\mathrm{Re}(a_{11}+a_{22})\leq 0$, $4\,\mathrm{Re}\,a_{11}\,\mathrm{Re}\,a_{22}-|a_{12}+\bar a_{21}|^2\geq 0$, and condition (5.9) — to inequalities

$$a_{11}+\bar a_{11}+2G<0,\quad (a_{11}+\bar a_{11}+2G)(a_{22}+\bar a_{22}+2G)-|a_{12}+\bar a_{21}|^2>0,$$

equivalent to inequalities $\mathrm{Re}(a_{11}+a_{22})+2G<0$,

$$4(\mathrm{Re}\,a_{11}+G)(\mathrm{Re}\,a_{22}+G)-|a_{12}+\bar a_{21}|^2>0.$$

During application of canonical expansion of unmodulated structure of the first form, the system of equations relative to canonical component for equation of oscillations of the second order has the form (see § 3, Chapter II)

$$\left.\begin{aligned}\dot y_1&=\left(\lambda_1-\frac{\lambda_1}{\lambda_1-\lambda_2}\right)y_1-\frac{\lambda_2}{\lambda_1-\lambda_2}y_2,\\ \dot y_2&=\frac{\lambda_1}{\lambda_1-\lambda_2}y_1+\left(\lambda_2+\frac{\lambda_2}{\lambda_1-\lambda_2}\right)y_2.\end{aligned}\right\}$$

Formulas for coefficients a_{ij} we obtain in the form

$$a_{11}=\lambda_1-\frac{\dot\lambda_1}{\lambda_1-\lambda_2},\quad a_{12}=-\frac{\dot\lambda_2}{\lambda_1-\lambda_2},$$
$$a_{21}=\frac{\dot\lambda_1}{\lambda_1-\lambda_2},\quad a_{22}=\lambda_2+\frac{\dot\lambda_2}{\lambda_1-\lambda_2}.$$

Condition of applicability of the considered canonical expansion has the form

$$b_1^2-4b_2\neq 0.$$

Given condition allows us to divide all equations of oscillations, for which it is executed, into two classes:

a) a class of equations satisfying condition

$$b_1^2-4b_2>0\qquad (t\geq t_0);$$

b) a class of equations satisfying condition

$$b_1^2-4b_2<0\qquad (t\geq t_0).$$

Roots λ_1 and λ_2 of equation (2.0) for systems of the first class during $t \geq t_0$ are real, for systems of second class they are complex conjugate. Therefore, when considering systems of the first class, we will talk about the "case of real roots;" when considering systems of the second class - about the "case of complex conjugate roots."

In the case of real roots, sufficient condition of stability takes the form

$$b_1+\frac{b_1\dot b_1-2\dot b_2}{b_1^2-4b_2}\geq 0,\quad b_2+\frac{8\dot b_1 b_2-4b_1\dot b_2-\dot b_1^2}{4(b_1^2-4b_2)}\geq 0$$
$$(t>T),$$

where T may be any amount large.

This condition is turned into sufficient condition of asymptotic stability if shown inequalities are replaced by inequalities

$$b_1+\frac{b_1\dot b_1+2\dot b_2}{b_1^2-4b_2}>2G,$$
$$b_2-b_1G+G^2+\frac{8\dot b_1 b_2-4b_1\dot b_2-\dot b_1^2-4G(b_1\dot b_1-2\dot b_2)}{b_1^2-4b_2}>0.$$

In the case of complex conjugate roots $\lambda_{1,2} = -a \pm i\omega$ ($i = \sqrt{-1}$) coefficients a_{jk} have the form

$$a_{11}=-a-\frac{\dot\omega}{2\omega},\quad a_{12}=\frac{\dot\omega-i\dot a}{2\omega},$$
$$a_{21}=\frac{\dot\omega+i\dot a}{2\omega},\quad a_{22}=-a-\frac{\dot\omega}{2\omega}.$$

Sufficient condition of stability in this case will be

$$a+\frac{\dot\omega}{2\omega}-\frac{\sqrt{\dot a^2+\dot\omega^2}}{2\omega}\geq 0\quad (t>T),$$

where it is turned into a sufficient condition of asymptotic stability if this inequality is replaced by inequality

$$\alpha + \frac{\dot{\omega}}{2\omega} - \frac{\sqrt{\dot{\alpha}^2 + \dot{\omega}^2}}{2\omega} > 0 \quad (t > T).$$

Expressing functions $\alpha(t)$ and $\omega(t)$ through coefficients of equation of oscillations, we will obtain sufficient condition of stability

$$b_1 + \frac{b_1\dot{b}_1 - 2\dot{b}_2}{b_1^2 - 4b_2} \geqslant \frac{2\sqrt{\dot{b}_1^2 b_2 - b_1\dot{b}_1\dot{b}_2 + \dot{b}_2^2}}{4b_2 - b_1^2} \qquad (t > T)$$

and sufficient condition of asymptotic stability

$$b_1 + \frac{b_1\dot{b}_1 - 2\dot{b}_2}{b_1^2 - 4b_2} - \frac{2\sqrt{\dot{b}_1^2 b_2 - b_1\dot{b}_1\dot{b}_2 + \dot{b}_2^2}}{4b_2 - b_1^2} > 0 \quad (t > T)$$

Analogously, can be obtained sufficient conditions of stability and asymptotic stability of oscillations, presented by second order equation, considering canonical expansions of first form of modified structure and canonical expansions of second form. In all cases these conditions can be expressed by inequalities connecting coefficients of equation of oscillations and their derivatives (first and highest).

In all inequalities, both in those which are connected with obtained-above conditions of stability and in those which pertain to other possible conditions of stability, it is possible to be liberated from irrationalities, considering (lost when raising to square their left and right sides) requirements for signs of the latter by additional inequalities. As a result of these operations, sufficient conditions of stability and asymptotic stability it is possible to express by conditions of nonnegativeness or positivity of a certain number of rational functions of coefficients of equation of oscillations and their derivatives.

Thus, for instance, the last of the obtained-above conditions of stability (for the case of complex conjugate roots $\lambda_{1,2}$) it is possible to express by two inequalities:

$$b_1 + \frac{\dot{c}}{2c} > 0; \qquad b_1^2 c + b_1\dot{c} - \dot{b}_1^2 \geqslant 0;$$

where $c = 4b_2 - b_1^2$.

We will return to conditions of stability and asymptotic stability (5.6) and (5.7).

Of interest is the case when certain canonical expansion of a solution reduces equation (0.1) to a system of equations relative to canonical components, between the coefficients of which there exist the following relationships:

a) $\mathrm{Re}\,\zeta_m^{(l-1)} \geq \mathrm{Re}\,\zeta_j^{(l-1)}$

($j = 1, \ldots, n$) for one or a larger number of values of index m during sufficiently large values of t;

b) at $t \to \infty$

$$A_{jk}^{(l)} = o(\mathrm{Re}\,\zeta_m^{(l-1)}) \qquad (j, k = 1, \ldots, n)$$

[or $g_{jk} = o(\mathrm{Re}\,\lambda_m)$ $(j, k = 1, \ldots, n)$, if $l = 1$ and is considered canonical expansion of first form].[1]

In this case, characteristic number μ_n is connected with magnitude $\mathrm{Re}\,\zeta_m^{(l-1)}$ by dependence

$$\mu_n = \mathrm{Re}\,\zeta_m^{(l-1)} + o(\mathrm{Re}\,\zeta_m^{(l-1)}),$$

where there exists such value $t = T$, starting from which magnitude μ_n does not change sign.

Actually, the validity of the last affirmation directly follows from conditions (a) and (b) and will become evident if we establish asymptotic form of Hermitian form

$$\frac{1}{2}\sum_{i,j=1}^{n}\left(2\mathrm{Re}\,\zeta_i^{(l-1)}\delta_{ij} + \bar{A}_{ij}^{(l-1)} + A_{ji}^{(l-1)}\right)e_i\bar{e}_j$$

and consider condition of its maximum. We can see the validity of the first affirmation if we present matrix

$$\left\|\begin{matrix} \mathrm{Re}\,\zeta_1^{(l-1)} + \frac{A_{11}^{(l)}+\bar{A}_{11}^{(l)}}{2} & \frac{A_{12}^{(l)}+\bar{A}_{21}^{(l)}}{2} & \ldots & \frac{A_{1n}^{(l)}+\bar{A}_{n1}^{(l)}}{2} \\ \frac{\bar{A}_{12}^{(l)}+A_{21}^{(l)}}{2} & \mathrm{Re}\,\zeta_2^{(l-1)} + \frac{A_{22}^{(l)}+\bar{A}_{22}^{(l)}}{2} & \ldots & \frac{A_{2n}^{(l)}+\bar{A}_{n2}^{(l)}}{2} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\bar{A}_{1n}^{(l)}+A_{n1}^{(l)}}{2} & \frac{\bar{A}_{2n}^{(l)}+A_{n2}^{(l)}}{2} & \ldots & \mathrm{Re}\,\zeta_n^{(l-1)} + \frac{A_{nn}^{(l)}+\bar{A}_{nn}^{(l)}}{2} \end{matrix}\right\|$$

in the form of the sum of two matrices:

$$\left\|\begin{matrix} \mathrm{Re}\,\zeta_1^{(l-1)} & 0 & 0 & \ldots & 0 \\ 0 & \mathrm{Re}\,\zeta_2^{(l-1)} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \mathrm{Re}\,\zeta_{n-1}^{(l-1)} & 0 \\ 0 & \ldots & 0 & 0 & \mathrm{Re}\,\zeta_n^{(l-1)} \end{matrix}\right\| + \left\|\begin{matrix} \frac{A_{11}^{(l)}+\bar{A}_{11}^{(l)}}{2} & \ldots & \frac{A_{1n}^{(l)}+\bar{A}_{n1}^{(l)}}{2} \\ \ldots & \ldots & \ldots \\ \frac{A_{n1}^{(l)}+\bar{A}_{1n}^{(l)}}{2} & \ldots & \frac{A_{nn}^{(l)}+\bar{A}_{nn}^{(l)}}{2} \end{matrix}\right\|$$

Maximum characteristic number of the first matrix is equal to $\mathrm{Re}\,\zeta_m^{(l-1)}$. Maximum characteristic number of the second matrix does not exceed the sum of moduli

---

[1] $f(t) = o[g(t)]$ at $t \to \infty$ means that $\frac{f(t)}{g(t)} \to 0$ at $t \to \infty$.

of its coefficients[1]

$$\sum_{i,j=1}^{n} |h_{ij}^{(l)}|.$$

Consequently, characteristic number μ_n, not exceeding sum of characteristic numbers of component matrices,[2] is not more than magnitude

$$\operatorname{Re}\zeta_m^{(l-1)} + \sum_{j,k=1}^{n} |h_{jk}^{(l)}|$$

and not less than the difference of its components. Expressed affirmation follows from this.

During application of canonical expansion of the first form, proof is analogous.

Because of shown dependence between magnitudes μ_n and Re $\zeta_m^{(l-1)}$, establishing their asymptotic equivalence, their integrals also are asymptotically equivalent, i.e.,

$$\lim_{t\to\infty} \frac{\int_{t_0}^{t} \operatorname{Re}\zeta_m^{(l-1)}\,dt}{\int_{t_0}^{t} \mu_n\,dt} = 1.$$

Hence, on the basis of equation (5.5), follows, at $t \to \infty$

$$|x(t)| \leqslant \sqrt{n}\, r_0 \left(\exp \int_{t_0}^{t} \operatorname{Re}\zeta_m^{(l-1)}\,dt\right)^{1+o(1)}.$$

Since magnitude Re $\zeta_m^{(l-1)}$, in a sufficiently remote interval (T, ∞), does not change sign, then, because of last relationship for stability, it is sufficient that there is executed condition

$$\operatorname{Re}\zeta_m^{(l-1)}(t) < 0 \qquad (T < t < \infty). \tag{5.10}$$

---

[1]At arbitrary fixed t, maximum characteristic number of this matrix is equal to maximum value of corresponding Hermitian form

$$\frac{1}{2}\sum_{i,j=1}^{n} (h_{ij}^{(l)} e_i \bar{e}_j + \bar{h}_{ij}^{(l)} \bar{e}_i e_j)$$

during condition $\sum_{i=1}^{n} e_i \bar{e}_i = 1$. Shown appraisal we will obtain after replacing coefficients $h_{ij}^{(l)}$ by their moduli and assuming $e_i = e_j = 1$.

[2]This follows from the fact that maximum value of the sum of corresponding Hermitian forms during condition $\sum_{i=1}^{n} e_i \bar{e}_i = 1$ cannot exceed the sum of maximum values of each component, determined under this condition.

Inasmuch as, according to condition (a), real parts of roots $\zeta_{i(i+m)}^{(l-1)}$ do not exceed real part of root $\zeta_m^{(l-1)}$, the last inequality is used if and only if coefficients of equation

$$(\zeta^{(l-1)})^n + b_1^{(l-1)}(\zeta^{(l-1)})^{n-1} + \ldots + b_n^{(l-1)} = 0 \qquad (5.11)$$

during any fixed $t = T$ satisfy conditions of Routh (see § 5 of this chapter) or Hurwitz.

The last obtained result we will formulate in the form of the following theorem.

Theorem. If a certain canonical expansion of solution leads equation (0.1) to such a system of equations relative to canonical components, at which are executed above-indicated conditions (a) and (b), then for stability of oscillations it is sufficient that coefficients of equation (5.11) at any fixed t, exceeding certain value $t = T$, satisfy conditions of Hurwitz or Routh.

In order, with the help of a fixed appraisal of the modulus of a particular solution, to reveal asymptotic stability, obviously one should replace condition (5.10) by condition

$$\lim_{t\to\infty}\int_{t_0}^{t} \operatorname{Re}\zeta_m\, dt = -\infty. \qquad (5.12)$$

Thus, the following theorem is valid.

Theorem. If there are executed conditions of the preceding theorem and condition (5.12), then oscillations are stable asymptotically.

Example: We will define sufficient conditions of stability of oscillations, presented by equation[1] (4.41):

$$\ddot{x} + ct^{\nu}x = 0, \quad c \neq 0, \quad \nu > -2,$$

using system of equations, obtained as a result of second canonical expansion. We will be limited by case $c > 0$.

Coefficients $\zeta_{1,2}^{(1)}$, $h_{11}^{(2)}$, $h_{22}^{(2)}$, $h_{21}^{(2)}$ of system (2.46), for considered case, were obtained in § 2, Chapter IV. In accordance with formulas for coefficients of this system

$$h_{12}^{(2)} = -h_{22}^{(2)}, \quad h_{21}^{(2)} = -h_{11}^{(2)},$$

during sufficiently large values of t, roots $\zeta_1^{(1)}$ and $\zeta_2^{(1)}$ are complex conjugate and coefficients $h_{jk}^{(2)}$ are connected by dependence

$$h_{11}^{(2)} = \bar{h}_{22}^{(2)}, \quad h_{12}^{(2)} = \bar{h}_{21}^{(2)}.$$

[1] See example in § 2, Chapter IV.

Using shown dependences, we will obtain an equation for determination of characteristic numbers of matrix

$$\left\| \frac{a_{ij}+\bar{a}_{ji}}{2} \right\|_1^2$$

in the form

$$\mu^2 - (\zeta_1^{(1)}+\zeta_2^{(1)}+A_{11}^{(2)}+A_{22}^{(2)})\mu + \frac{(A_{11}^{(2)}-A_{22}^{(2)})^2+(\zeta_1^{(1)}+\zeta_2^{(1)})^2}{4} + \frac{(\zeta_1^{(1)}+\zeta_2^{(1)})(A_{11}^{(2)}+A_{22}^{(2)})}{2} = 0,$$

or

$$\mu^2 + \frac{1}{2}\frac{d\ln\theta}{dt}\mu + \frac{\nu^2}{16t^2}\left[\frac{\left(\frac{\nu}{2}+1\right)^2}{\theta t^2} - 1\right] + \frac{\nu}{8t}\frac{d\ln\theta}{dt} = 0,$$

where

$$\theta = \frac{\nu^2}{4t^2} - 4ct^{\nu}.$$

For roots of this equation $\mu_{1,2}$ we will obtain formula

$$\mu_{1,2} = -\frac{1}{4}\frac{d\ln\theta}{dt} \pm \frac{1}{4}\sqrt{\left(\frac{d\ln\theta}{dt}\right)^2 - \frac{2\nu}{t}\frac{d\ln\theta}{dt} - \frac{\nu^2}{t^2}\left[\frac{\left(\frac{\nu}{2}+1\right)^2}{\theta t^2} - 1\right]}.$$

At $t \to \infty$

$$\mu_{1,2} = -\frac{\nu}{4t} + O(t^{-1-\alpha}),$$

where α is a certain positive constant. Integrating this expression, we will find

$$\int \mu_{1,2}\,dt = -\frac{\nu}{4}\ln t + O(t^{-\alpha}) + \text{const}.$$

It is easy to see that

$$\lim_{t\to\infty}\int \mu_{1,2}\,dt = \infty \quad \text{when} \quad \nu < 0,$$
$$\lim_{t\to\infty}\int \mu_{1,2}\,dt = -\infty \quad \text{when} \quad \nu > 0.$$

It follows from this, that for asymptotic stability of oscillations presented by equation (4.41), at $c > 0$ it is sufficient to execute condition

$$\nu > 0.$$

Case $\nu = 0$ is a trivial case of harmonic oscillations, stable but not asymptotic.

§ 3. Sufficient Conditions of Stability, Based on Properties of Weighted Canonical Components

In the preceding paragraph we established sufficient conditions of stability and asymptotic stability of oscillations, using majorant appraisal of norm of solution of system of equations relative to canonical components. In this paragraph we will establish analogous conditions, taking for a basis majorant appraisal of norm of solution of system of equations relative to weighted canonical components.

We will assume that in certain interval (t_0, ∞) are executed conditions of applicability of k-th canonical expansion of second form. For the purpose of simplification, we will consider only expansion of an unmodulated structure.

As was shown in § 4 of preceding chapter, solution of equation of oscillations (0.1) is connected with norm of solution of system of equations relative to weighted canonical components R(t) by dependence (4.66)

$$x = R \sum_{i=1}^{n} \frac{\tilde{x}_i E_i}{C_i}.$$

where E_i $(i = 1, \ldots, n)$ are phase coefficients of solution of mentioned system, and $\tilde{x}_i/C_i$ are approximate solutions of equation of oscillations, determined by equalities (4.23)

$$\frac{\tilde{x}_i}{C_i} = \exp \int_{t_0}^{t} \zeta_i^{(k-1)}\, dt$$

or (4.24)

$$\frac{\tilde{x}_i}{C_i} = \exp \int_{t_0}^{t} (\zeta_i^{(k-1)} + h_{ii}^{(k)})\, dt.$$

From equality (4.66) it follows that for stability of oscillations it is sufficient that magnitude

$$R \frac{\tilde{x}_i}{C_i} \qquad (i = 1, \ldots, n)$$

be limited in interval (t_0, ∞) by functions t, and for asymptotic stability it is sufficient that they approach zero at $t \to \infty$. Obviously, shown magnitudes will be limited or vanishing functions if these properties, correspondingly, possess magnitudes

$$\mathrm{Maj}\, R \frac{\tilde{x}_i}{C_i} \qquad (i = 1, \ldots, n).$$

Using equalities (4.23) or (4.24) and first equality of system (4.58), connecting majorant of norm of solution of system of equations relative to weighted canonical component with the biggest characteristic number M_n of its corresponding Hermitian matrix, we will find that for stability of oscillations it is sufficient that there are executed inequalities

$$\overline{\lim_{t \to \infty}} \int_{t_0}^{t} (M_n + \mathrm{Re}\, \zeta_i^{(k-1)})\, dt < \infty \quad (i = 1, \ldots, n). \tag{5.6'}$$

and for asymptotic stability, it is sufficient to fulfill equalities

$$\lim_{t \to \infty} \int_{t_0}^{t} (M_n + \mathrm{Re}\, \zeta_i^{(k-1)})\, dt = -\infty \quad (i = 1, \ldots, n). \tag{5.7'}$$

Obviously, stability will take place if in certain interval (T, ∞), where $T > t_0$, there is executed condition

$$M_n + \operatorname{Re}\zeta_i < 0 \quad (i = 1, \ldots, n). \tag{5.8'}$$

which is less effective, but more convenient in application than condition (5.[illegible]). Analogously, for asymptotic stability it is sufficient to fulfill, in interval (T, ∞), conditions

$$M_n + \operatorname{Re}\zeta_i < G < 0 \quad (i = 1, \ldots, n;\ G = \text{const}). \tag{5.9'}$$

which is in the same relation to condition (5.7').

§ 4. Minorant Appraisal of Moduli of Particular Solutions of an Equation of Oscillations

In § 2 during determination of sufficient conditions of stability there was used appriasal (5.5), limiting from above growth of modulus of any particular solution of an equation of oscillations. Obviously, it would have been possible to obtain necessary conditions of stability if we managed to establish appraisals limiting growth of moduli of particular solutions from below. Since the idea of instability is connected with the existence of a certain particular solution possessing a definite property (unlimitedness), we, by no means, are interested in obtaining such minorant appraisals which are applicable to all nontrivial particular solutions; the narrower the class of particular solutions for which are true these or other minorant appraisals, the stronger and more necessary the conditions of stability to which they lead.

This prompts the requirement which we will hold during establishment of minorant appraisals of moduli of particular solutions: there have to exist particular solutions to which these appraisals are applicable.

The problem of determining minorant appraisals of moduli of particular solutions is very complicated, as a consequence of which we cannot rely on its solution in a general equation of oscillations. The results obtained here are not applicable to all equations of oscillations.

Assuming that coefficients of an equation of free oscillations are a sufficient number of times differentiable functions, we will consider system of equations relative to phase coefficients, obtained as a result of l-th canonical expansion of mentioned solution by components $z_1, \ldots, z_n$. In accordance with equations (3.15), we will write it in the form

$$\dot{c}_i = (\zeta_i^{(l-1)} - E)c_i + \sum_{j=1}^{n} h_{ij}^{(l)} c_j \quad (i = 1, \ldots, n). \qquad (5.1[illegible])$$

where F is Hermitian form of variables $e_1, \ldots, e_n$, corresponding to quadratic form F of variables $f_1, \ldots, f_n$.

Let us assume that m is index of that from roots $\zeta_i^{(l-1)}$ ($i = 1, \ldots, n$), which has the biggest real part, i.e.,

$$\operatorname{Re}\zeta_m^{(l-1)} > \operatorname{Re}\zeta_i^{(l-1)} \quad (i = 1, \ldots, n). \tag{5.14}$$

Let us consider two cases:

a) $\operatorname{Im}\zeta_m^{(l-1)} = 0$, inequality (5.14) is executed only for one value of m,

b) $\operatorname{Im}\zeta_m^{(l-1)} \neq 0$, inequality (5.14) is executed for two values of m.[1]

Let us assume that during certain t = T, phase coefficients at $i \neq m$ satisfy inequalities

$$|e_i| < \varepsilon < 1. \tag{5.15}$$

Then for phase coefficients e_m in the first case

$$|e_m|^2 = 1 - \sum_{\substack{i=1 \\ i \neq m}}^{n} |e_i|^2 > 1 - (n-1)\varepsilon^2,$$

in the second case

$$|e_{m_1}|^2 + |e_{m_2}|^2 > 1 - (n-2)\varepsilon^2,$$

where m_1 and m_2 are value of indices m.

We will assume further that at $t \to \infty$

$$\left.\begin{aligned} \operatorname{Re}\zeta_i^{(l-1)} &= O(\operatorname{Re}\zeta_m^{(l-1)}) \quad (i \neq m), \\ \operatorname{Re}\zeta_m^{(l-1)} &= O(\operatorname{Re}\zeta_m^{(l-1)} - \operatorname{Re}\zeta_i^{(l-1)}) \quad (i \neq m), \\ h_{ij}^{(l)} &= o(\operatorname{Re}\zeta_m^{(l-1)}). \end{aligned}\right\} \tag{5.16}$$

In accordance with conditions (5.15) it is always possible to indicate such value t = T that at t ≥ T there will be executed conditions

$$\left.\begin{aligned} |\operatorname{Re}\zeta_i^{(l-1)}| &< p\,|\operatorname{Re}\zeta_m^{(l-1)}| \quad (i \neq m), \\ |\operatorname{Re}\zeta_m^{(l-1)}| &< q\,(\operatorname{Re}\zeta_m^{(l-1)} - \operatorname{Re}\zeta_i^{(l-1)}) \quad (i \neq m), \\ |h_{ij}^{(l)}| &< \delta(t)\,|\operatorname{Re}\zeta_m^{(l-1)}|, \end{aligned}\right\} \tag{5.17}$$

where p and q are certain positive constants; δ(t) is positive, monotonic vanishing function.

We will consider that selection of magnitude T is subject to this condition.

We will estimate from below Hermitian form

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[1] Corresponding indices, obviously, must belong to complex-conjugate roots.

$$E=\frac{1}{2}\sum_{i=1}^{n}\left[(\zeta_i^{(l-1)}+\bar{\zeta}_i^{(l-1)})e_i\bar{e}_i-\bar{e}_i\sum_{j=1}^{n}h_{ij}^{(l)}e_j+e_i\sum_{j=1}^{n}\bar{h}_{ij}^{(l)}\bar{e}_j\right].$$

Using inequalities (5.15) and (5.17), we will obtain for t = T in case (a)

$$E\geqslant \operatorname{Re}\zeta_m^{(l-1)}-[(n-1)\varepsilon^2+(n-1)p\varepsilon^2+n^2\delta(t)]\,|\operatorname{Re}\zeta_m^{(l-1)}|. \quad (5.18)$$

in case (b)

$$E\geqslant \operatorname{Re}\zeta_m^{(l-1)}-[(n-2)\varepsilon^2+(n-2)p\varepsilon^2+n^2\delta(t)]\,|\operatorname{Re}\zeta_m^{(l-1)}|.$$

Since from the second inequality follows the first (during identical values of analogous magnitudes), subsequently we will consider only the first inequality.

Inequality (5.18), as is easy to see, so connects magnitudes E and Re $\zeta_m^{(l-1)}$ that at $\varepsilon \to 0$ and $t = T \to \infty$

$$\frac{E-\operatorname{Re}\zeta_m^{(l-1)}}{\operatorname{Re}\zeta_m^{(l-1)}}\to 0.$$

During $t \geq T$ inequality (5.18) remains in force as long as there will not be disturbed at least one of the inequalities (5.15). From differentiability of phase coefficients, it follows that if inequality (5.15) is executed at t = T, then it also will be executed in a certain finite section $[T, T+\tau]$. Because of that noted above, in this section also should be executed inequality (5.18).

We will designate the lower boundary of form E, established by inequality (5.18), by symbol $\underline{E}$. Obviously, difference Re $\zeta_m^{(l-1)} - \underline{E}$ is always positive.

Due to inequalities standing in second line of system (5.17) and inequality (5.18), moment of time T it is always possible to select in such a way that in finite section $[T, T+\tau]$ for solutions satisfying conditions (5.15) in this section during sufficiently small values ε in addition to inequalities (5.17), there are executed inequalities

$$\frac{\operatorname{Re}\zeta_m^{(l-1)}-\operatorname{Re}\zeta_i^{(l-1)}}{\underline{E}-\operatorname{Re}\zeta_i^{(l-1)}}<s, \quad (5.19)$$

where s is a certain positive constant, and so that with this difference $\underline{E}$ - Re $\zeta_i^{(l-1)}$ $(i \neq m)$ are positive.

Actually, we will present difference $\underline{E}$ - Re $\zeta_m^{(l-1)}$ in the form of the sum of two components:

$$\operatorname{Re}\zeta_m^{(l-1)}-\operatorname{Re}\zeta_i^{(l-1)} \text{ and } \underline{E}-\operatorname{Re}\zeta_m^{(l-1)}.$$

On the basis of inequalities standing in the second line of system (5.17), for sufficiently large t we have

$$\operatorname{Re}\zeta_m^{(l-1)}-\operatorname{Re}\zeta_i^{(l-1)}>\frac{1}{q}\,|\operatorname{Re}\zeta_m^{(l-1)}|.$$

where q is a certain positive constant. In accordance with inequality (5.18), by definition of magnitude $\underline{E}$ for sufficiently large value of t, for which are satisfied conditions (5.15), and in a certain finite interval, adjoining it on the right, we have

$$\underline{E} - \operatorname{Re} \zeta_m^{(l-1)} = -|(n-1)(p+1)\varepsilon^2 + n^2\delta|\,|\operatorname{Re} \zeta_m^{(l-1)}|,$$

where p is a certain positive constant; $\delta = \delta(t)$ is a positive, monotonic, vanishing function.

Obviously, it is always possible to select such value $t = T$ and such value $\varepsilon = \varepsilon_0$ that during all $\varepsilon \leq \varepsilon_0$ in section $[T, T + \tau]$ there will be executed inequality

$$\frac{1}{q} > |(n-1)(p+1)\varepsilon^2 + n^2\delta|, \tag{5.20}$$

at which all differences $\underline{E}$ - Re $\zeta_i^{(l-1)}$ $(i \neq m)$ are positive.

Assuming that selection of magnitudes T and ε is subject to the shown, additional conditions, we will estimate fractions

$$\frac{\operatorname{Re} \zeta_m^{(l-1)} - \operatorname{Re} \zeta_i^{(l-1)}}{\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}} \qquad \begin{pmatrix} T \leq t \leq T + \tau \\ i \neq m \end{pmatrix}.$$

In the same conditions with which preceding proof is carried out, we will obtain

$$\frac{\operatorname{Re} \zeta_m^{(l-1)} - \operatorname{Re} \zeta_i^{(l-1)}}{\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}} = \frac{\operatorname{Re} \zeta_m^{(l-1)} - \operatorname{Re} \zeta_i^{(l-1)}}{\operatorname{Re} \zeta_m^{(l-1)} - \operatorname{Re} \zeta_i^{(l-1)} + \underline{E} - \operatorname{Re} \zeta_m^{(l-1)}} =$$
$$= \frac{\operatorname{Re} \zeta_m^{(l-1)} - \operatorname{Re} \zeta_i^{(l-1)}}{\operatorname{Re} \zeta_m^{(l-1)} - \operatorname{Re} \zeta_i^{(l-1)} - |(n-1)(p+1)\varepsilon^2 + n^2\delta|\,|\operatorname{Re} \zeta_m^{(l-1)}|} <$$
$$< \frac{1}{1 - q\,|(n-1)(p+1)\varepsilon^2 + n^2\delta|}.$$

Obviously, from inequality (5.20) ensues inequality

$$0 < \frac{1}{1 - q\,|(n-1)(p+1)\varepsilon^2 + n^2\delta|} < s,$$

where s is a certain positive constant. Hence, due to preceding inequality, there follows inequality (5.19).

Assuming that conditions connected with inequalities (5.15) (5.17), and (5.20) are executed, we will estimate behavior of phase coefficients in section $[T, T + \tau]$.

Multiplying term by term i-th $(i \neq m)$ equation of the system (5.13) by $\overline{e}_i$ and adding left and right side of the new equation with corresponding parts of equation obtained from the latter by means of transition to conjugate complex numbers we will obtain equation

$$\frac{1}{2}\frac{d}{dt}|e_i|^2 = (\operatorname{Re} \zeta_i^{(l-1)} - \underline{E})|e_i|^2 + \frac{1}{2}\sum_{j=1}^{n} (\overline{h}_{ij}^{(l)} e_i \overline{e}_j + h_{ij}^{(l)} \overline{e}_i e_j).$$

Using inequalities (5.17) and (5.18), this equality it is possible to replace by inequality

$$\frac{1}{2}\frac{d}{dt}|e_i|^2 < (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})|e_i|^2 + n\delta qs(\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})|e_i|,$$

which, obviously is equivalent to inequality

$$\frac{d}{dt}|e_i| < (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})|e_i| + n\delta qs(\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)}). \tag{5.21}$$

Inequality (5.21) it is possible to replace by equivalent equation

$$\frac{d}{dt}|e_i| = (\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})|e_i| + nqs\delta(\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)}) - k_i(t), \tag{5.22}$$

where $k_1(t)$ is a certain positive function.

Particular solution of equation (5.22), describing behavior of i-th phase coefficient, is found by the formula

$$|e_i| = \exp\int_T^t (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\,dt \left\{\int_T^t [n\delta qs(\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)}) - k_i] \times \right.$$
$$\left. \times \exp\int_T^t (\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})\,dt\,dt + |e_i(T)|\right\}$$

and, consequently, consists of three components:

$$|e_i(T)|\exp\int_T^t (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\,dt,$$

$$-\exp\int_T^t (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\,dt \int_T^t k_i \exp\int_T^t (\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})\,dt\,dt,$$

$$nqs\delta\exp\int_T^t (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\,dt \int_T^t \delta(\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})\exp\int_T^t (\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})\,dt\,dt.$$

The first component is a monotonically diminishing function. The second component is equal to zero when t = T and negative for t > T. The third component is equal to zero when t = T and positive for t > T.

Obviously, subsequent values of magnitude $|e_i|$ can exceed its initial value $|e_i(T)|$ only because of change of the third component. In order to clarify this possibility, we will compare first derivatives of first and third components.

First derivative of first component, obviously, is magnitude

$$|e_i(T)|(\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\exp\int_T^t (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\,dt.$$

First derivative of the third component is equal to magnitude

$$nqs(\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\exp\int_T^t (\mathrm{Re}\,\zeta_i^{(l-1)} - \underline{E})\,dt \int_T^t (\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})\delta \times$$
$$\times \exp\int_T^t (\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)})\,dt\,dt + n\delta qs(\underline{E} - \mathrm{Re}\,\zeta_i^{(l-1)}).$$

Because of inequality

$$\int_{T}^{t} (\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}) \delta \exp \int_{T}^{t} (\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}) \, d\tau \, dt =$$

$$\int_{T}^{t} \delta d \left[\exp \int_{T}^{t} (\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}) \, d\tau \right] \leq \delta(T) \int_{T}^{t} d \left[\exp \int_{T}^{t} (\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}) \, d\tau \right] =$$

$$= \delta(T) \left[\exp \int_{T}^{t} (\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}) \, dt - 1 \right]$$

it is less than magnitude

$$n\delta(T) qs \, (\underline{E} - \operatorname{Re} \zeta_i^{(l-1)}) \exp \int_{T}^{t} (\operatorname{Re} \zeta_i^{(l-1)} - \underline{E}) \, dt.$$

We will limit, additionally, selection of magnitude $e_1(T)$ by inequality

$$|e_1(T)| \leqslant \varepsilon_1,$$

and selection of magnitude T by inequality

$$nqs\delta(T) < \varepsilon_1,$$

where ε_1 is a positive value, smaller than magnitude ε_0.

This is always possible.

We will set, arbitrarily, magnitude ε_1 and will consider solution for which

$$|e_1(T)| = \varepsilon_1. \tag{5.23}$$

Having compared derivatives of the first and third components, we will be convinced that for this solution, first magnitude numerically exceeds second during all $T \leq t$. Consequently, for solution satisfying condition (5.23), $|e_1(t)|$ is a diminishing function. It follows from this that at the end of section $[T, T + \tau]$ there occurs inequality

$$|e_1(T + \tau)| < \varepsilon_1.$$

Let us assume that $|e_1(t)|^*$ is majorant of function $|e_1(t)|$, satisfying condition (5.23), fixed by means of replacing the derivative of the third component by its above-indicated upper appraisal. Then for all solutions satisfying condition

$$|e_1(T)| < \varepsilon_1, \tag{5.24}$$

there occurs inequality

$$|e_1(T + \tau)| < |e_1(T + \tau)|^*. \tag{5.25}$$

The justice of this inequality is seen by the following reasoning.

During condition (5.24) majorant of function $|e_1(t)|$, found by the same method, can be increasing in a certain interval, adjoining, on the right, to point $t = T$. If its growth does not change relationship

$$\mathrm{Maj}\,|e_i(t)| < |e_i(t)|^*.$$

then condition (5.25) is executed. If, however, at a certain point there is established equality

$$\mathrm{Maj}\,|e_i(t)| = |e_i(t)|^*.$$

then, taking this value of t for a new T, we will be convinced by the above-mentioned formulas that during further change of t majorants of comparable functions coincide. Consequently, condition (5.25) is executed also in this case.

Thus, it is proven that for all solutions, i-th phase coefficient of which satisfies condition

$$|e_i(T)| \leqslant \varepsilon_1 < \varepsilon_0,$$

independently of length of section [T, T + τ]

$$|e_i(t)| < \varepsilon_1 \qquad (T < t \leqslant T + \tau).$$

This conclusion is valid for all phase coefficients with indices $i \neq m$.

Revealed property of phase coefficients $e_i (i \neq m)$ leads us to the following conclusion.

<u>If magnitudes ε and T are selected properly (as above-indicated), then for all $i \neq m$ from condition</u>

$$|e_i(T)| \leqslant \varepsilon_1 < \varepsilon_0$$

<u>follows</u>

$$|e_i(t)| < \varepsilon_1$$

<u>during all $t > T$</u>.

Really, if one were to assume that there exists a certain maximum final section [T, T + τ], on the right from which is disturbed last inequality, then it is possible to arrive at conclusion that at point $t = \tau$ one of functions $|e_i(t)|$ should be changed by a jump. But this contradicts property of continuity of phase coefficients.

Using noted property of phase coefficients, relationship

$$\frac{\rho}{r} = E$$

and the lower appraisal of form E, delivered by inequality (5.18), for solutions whose phase coefficients satisfy conditions

$$|e_i(T)| < \varepsilon_1 < \varepsilon_0 < 1 \quad (i \neq m).$$

during proper selection of magnitudes ε_0 and T, we will obtain the following lower appraisal of norm:

$$r \geqslant C \exp \int_{T}^{t} \underline{E}\, dt = C\left(\exp \int_{T}^{t} \operatorname{Re} \zeta_m^{(l-1)}(t)\, dt\right)^{1-\eta_1(T) \operatorname{sign} \operatorname{Re} \zeta_m^{(l-1)}}. \tag{5.26}$$

where C is a certain real positive constant.

$$\eta_1 = |(n-1)\varepsilon^2 + (n-1)p\varepsilon^2 + n^2\delta|.$$

Thus, there is fixed minorant appraisal of norms of certain particular solutions of the system of equations relative to canonical components.

Using obtained results and passing to an equation of free oscillations, we will establish corresponding minorant appraisals of moduli of their particular solutions. We will distinguish cases (a) and (b) shown in the beginning of the paragraph.

<u>In case (a)</u>, from inequality (5.26), because of relationship

$$|x| = r\left|\sum_{i=1}^{n} e_i\right|$$

and inequalities

$$|e_i| < \varepsilon \qquad (i \neq m),$$

in which ε is considered a sufficiently small magnitude, there follows

$$|x| > C(1-\eta_2)\left(\exp \int_{T}^{t} \operatorname{Re} \zeta_m^{(l-1)}\, dt\right)^{1-\eta_1 \operatorname{sign} \operatorname{Re} \zeta_m^{(l-1)}},$$

where $\eta_2 = (n-1)\varepsilon\left(1 + \frac{\varepsilon}{2}\right)$.

Considering $\eta_2 < 1$, and after changing designation $C(1 - \eta_2)$ to C and η_1 to η, the latter inequality we will copy in the form

$$|x| > C\left(\exp \int_{T}^{t} \operatorname{Re} \zeta_m^{(l-1)}\, dt\right)^{1-\eta \operatorname{sign} \operatorname{Re} \zeta_m^{(l-1)}}. \tag{5.27}$$

Magnitudes η_1 and η, appearing in inequalities (5.26) and (5.27), become as small as desired during $\varepsilon_0 \to 0$ and $T \to \infty$.

Obtained results it is possible to formulate in the form of the following theorem.

<u>Theorem.</u> <u>If equation of free oscillations allows l-th canonical expansion of the second form, where there are executed inequalities (5.14) for one</u> ($\operatorname{Im} \zeta_m^{(l-1)} = 0$) <u>value of m and conditions (5.16), then there exists particular solution of this equation, satisfying, during sufficiently large T, inequality (5.27) where η is a given positive value as small as desired beforehand, and C is an arbitrary positive constant.</u>

This theorem establishes sufficient conditions of the existence of particular solutions of equation of oscillations, possessing definite properties, without

indications of whether solutions are complex or real. It is necessary to note that during proof of the theorem, it is always possible to select initial values of phase coefficients so that their corresponding solutions of equation of oscillations are real. Therefore, among the solutions whose existence is stated in the theorem, there must be real solutions.

In case (b), transition from appraisal of function $r(t)$ to appraisal of moduli of solutions $x(t)$ is somewhat more complicated. This is explained by the fact that from inequality

$$|x| \geqslant r|e_{m_1}+e_{m_2}| - r\sum_{i=1}^{n}|e_i| \quad (i \neq m_1, i \neq m_2), \tag{5.28}$$

which connects magnitude $|x|$ and r, in this case it is impossible to make an appraisal analogous to appraisal (5.27), since magnitude $|e_{m_1} + e_{m_2}|$ in principle may be as small as desired. In order to establish an appraisal interesting us, here are necessary additional reasonings. These reasonings are given below. For simplification, it is assumed that initial values of phase coefficients, appearing during proof of appraisal (5.26), are such that their corresponding solutions of equation of oscillations are real. This assumption does not change the mentioned proof.

For real solutions $x(t)$, phase coefficients e_{m_1} and e_{m_2}, as all other pairs of phase coefficients corresponding to complex conjugate functions $\zeta_i(t)$ are also complex conjugate phase coefficients e_{i_1} and e_{i_2} in the form

$$e_{i_1} = \frac{1}{\sqrt{2}}(f_{i_1} + i f_{i_2}),$$
$$e_{i_2} = \frac{1}{\sqrt{2}}(f_{i_1} - i f_{i_2}),$$

where f_{i_1} and f_{i_2} are real magnitudes, $i = \sqrt{-1}$, after changing designation e_j to f_j for indices j corresponding to real functions ζ_j, we will replace system of equations (5.13) by system

$$\dot{f}_j = \sum_{k=1}^{n}(h'_{jk} - \delta_{jk}F)\, f_k \qquad (j = 1, \ldots, n), \tag{5.29}$$

where h'_{jk} are certain functions of coefficients $h_{jk}^{(l)}$ and $\zeta_j^{(l-1)}$ ($j, k = 1, \ldots, n$), F is quadratic form obtained from Hermitian form

$$\frac{1}{2}\sum_{i=1}^{n}\left[(\zeta_i^{(l-1)} + \bar{\zeta}_i^{(l-1)})\, e_i\bar{e}_i + \bar{e}_i\sum_{j=1}^{n} h_{ij}^{(l)} e_j + e_i\sum_{j=1}^{n}\bar{h}_{ij}^{(l)}\bar{e}_j\right]$$

after replacement of magnitudes e_i ($i = 1, \ldots, n$) by f_j ($j = 1, \ldots, n$) (see § 9, Chapter III).

We will set arbitrary initial values of variables f_i ($i = 1, \ldots, n$; $i \neq m_1$; $i \neq m_2$) in such a manner that there are executed inequalities

$$|e_i(T)| < \varepsilon_1,$$

where ε_1 is a magnitude introduced during conclusion of inequality (5.26).

With such initial values of above-indicated variables, initial values of functions f_{m_1} and f_{m_2} have to satisfy inequality

$$f^2_{m_1}(T) + f^2_{m_2}(T) > 1 - (n-2)\varepsilon_1^2.$$

Let us assume that c is any number satisfying condition

$$1 > c > 1 - (n-2)\varepsilon_1^2,$$

and $\{x(t)\}_c$ is a set of solutions $x(t)$ determined by equality

$$f^2_{m_1}(T) + f^2_{m_2}(T) = c \qquad (5.30)$$

during arbitrary fixed initial value of norm r. Then, on a plane with a rectangular system of coordinates, on the axes of which are values of variables f_{m_1} and f_{m_2}, locus of points given by equality (5.30) constitutes a circumference with the center at the origin of coordinates and the radius $\sqrt{c}$.

We will consider locus of points to which is transformed shown circumference during change of t to arbitrary value $t_1 > T$.

After applying the theorem of continuous dependence of the solution of a system of differential equations on initial data [22] to system (5.29), we will find that this locus is a closed curve. According to the above-proven [see conclusion of inequality (5.26)], for all points of this circle there should be satisfied inequality

$$f^2_{m_1}(t_1) + f^2_{m_2}(t_1) > 1 - (n-2)\varepsilon_1^2. \qquad (5.31)$$

After selecting solution of system (5.29), for which $f_{m_2}(t_1) = 0$ in accordance with inequalities (5.28) and (5.31) we will obtain

$$|x(t_1)| > r\sqrt{1-(n-2)\varepsilon_1^2} - r(n-2)\varepsilon_1.$$

If one were to consider only such values of t for which this inequality is executed, then, with these values of t for moduli of particular solutions of equation of oscillations corresponding to selected solution of system (5.29), we will obtain appraisal (5.27). Analogous appraisal during all other values of t we will obtain for moduli of particular solutions of equation of oscillations, corresponding to certain other solutions of this system. Thus, we will come to the conclusion which it is possible to formulate in the form of the following theorem.

Theorem. If in certain interval (t_0, ∞) an equation of free oscillations allows l-th canonical expansion of the second form, where are executed inequalities (5.14) for the two values m_1 and m_2 of index m and conditions (5.16), then there exist real particular solutions of this equations, such that during every $t > T$ one of them must satisfy inequality (5.27) where T, η, and C are constants, the meaning of which is shown in the preceding theorem.

In the beginning of this paragraph it was assumed that coefficients of an equation of free oscillations are (a sufficient number of times) differentiable functions, and there is applied to this equation transformation into a system of equations on the basis of canonical expansion of solution of the second form. If one were to be limited by the weaker assumption that coefficients of an equation of oscillations are at least singly differentiable functions, and to assume additionally that coefficients of equation (2.0) do not have multiple roots during any value of t from certain interval (t_0, ∞), then an equation of oscillations it is possible to convert into a system on the basis of canonical expansion of solution of the first form. To the system obtained as a result of this transformation can be applied completely all the above-mentioned reasoning. For that, it is necessary only that functions $\zeta_1^{(l-1)}(t), \ldots, \zeta_n^{(l-1)}(t)$ be replaced by functions $\lambda_1(t), \ldots, \lambda_n(t)$, and functions $h_{ij}^{(l)}(t)$ $(i, j = 1, \ldots, n)$ by functions $g_{ij}(t)$ $(i, j = 1, \ldots, n)$. These reasonings lead to the result presented in the following theorem.

Theorem. If in certain interval (t_0, ∞) coefficients of an equation of free oscillations are differentiable, roots of equation (2.0) are simple, there is executed condition

$$\begin{aligned} &\operatorname{Re}\dot{\lambda}_i = O(\operatorname{Re}\lambda_m) && (i \neq m), \\ &\operatorname{Re}\dot{\lambda}_m = O(\operatorname{Re}\lambda_m - \operatorname{Re}\lambda_i) && (i \neq m), \\ &g_{ij} = o(\operatorname{Re}\lambda_m) && (i, j = 1, \ldots, n) \end{aligned} \tag{5.32}$$

and inequalities

$$\operatorname{Re}\lambda_m \geq \operatorname{Re}\lambda_i \quad (i = 1, \ldots, n) \tag{5.33}$$

for one (if $\operatorname{Im}\lambda_m = 0$) or for two (if $\operatorname{Im}\lambda_m \neq 0$) values of m, then in the first case there exists a real particular solution of the first equation satisfying in interval

(T, ∞), during sufficiently large T, inequality

$$|x| > C\left(\exp\int_T^t \operatorname{Re}\lambda_m\,dt\right)^{1-\eta\,\operatorname{sign}\operatorname{Re}\lambda_m}. \tag{5.34}$$

where η is beforehand given positive constant as small as desired; C is arbitrary positive constant, and in the second case there exists a class of real particular solutions, such as that during every $t > T$ one of them must satisfy this inequality.

Example: We will define minorant appraisals of moduli of certain particular solutions of equation

$$\ddot{x} + ct^{\nu}x = 0, \quad c > 0, \quad \nu > -2,$$

using system of equations obtained as a result of second canonical expansion.

Coefficients $\zeta_{1,2}^{(1)}$; $h_{ij}^{(2)}$ (i, j = 1, 2) for that equation have the form

$$\zeta_{1,2}^{(1)} = -\frac{\nu}{4t} \mp \frac{1}{2}\sqrt{\frac{\nu^2}{4t^2} - 4ct^{\nu}},$$

$$h_{11}^{(2)} = -h_{21}^{(2)} = \frac{\nu}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-\frac{1}{2}} + \frac{\nu}{4t} - \frac{1}{4}\frac{d\ln\theta}{dt},$$

$$h_{22}^{(2)} = -h_{12}^{(2)} = -\frac{\nu}{4t^2}\left(\frac{\nu}{2}+1\right)\theta^{-\frac{1}{2}} + \frac{\nu}{4t} - \frac{1}{4}\frac{d\ln\theta}{dt},$$

where $\theta = \dfrac{\nu}{4t^2} - 4ct^{\nu}$ (see § 2, Chapter IV and § 2 of this chapter).

During sufficiently large values of t, roots $\zeta_1^{(1)}$ and $\zeta_2^{(1)}$ are complex; therefore, inequalities (5.14) are executed for two values of m (1 and 2).

The first two conditions of system (5.16) in this case are dropped since there do not exist indices i differing from m. The latter condition is executed during $\nu \neq 0$, since under this condition

$$|h_{ij}^{(2)}| = o\left(\frac{1}{t}\right) \quad (i, j = 1, 2),$$

$$\operatorname{Re}\zeta_{1,2}^{(1)} = -\frac{\nu}{4t}.$$

Consequently, during $\nu \neq 0$ there are carried out conditions of the second theorem. Applying it, we will find that during $\nu \neq 0$, for any positive values of magnitudes C and η it is always possible to indicate class of real particular solutions, taking during $t = t_0 > 0$ limited initial values $x(t_0)$ and $x(t_0)$, such that during sufficiently large value of T during every $t > T$, one of them obeys inequality

$$|x| > C\left[\exp\left(-\frac{\nu}{4}\right)\int_{\tau}^{t} t^{-1}\,dt\right]^{1+\eta\,\mathrm{sign}\,\nu} =$$

$$= \begin{cases} = C' t^{-\frac{\nu(1-\eta)}{4}} & \text{when } \nu < 0, \\ = C'' t^{-\frac{\nu(1+\eta)}{4}} & \text{when } \nu > 0. \end{cases}$$

§ 5. Necessary Conditions of Stability

Proven in the preceding paragraph, theorems establish conditions of existence of particular solutions satisfying inequalities (5.27) or (5.34) during any beforehand-given positive constants C and η. If oscillations are stable, then all particular solutions of equation (0.1) are limited. Consequently, if the conditions of one of the mentioned theorems are executed, then for stable oscillations, particular solutions satisfying, accordingly, inequalities (5.27) or (5.34) also have to be limited. Fixing arbitrary value C and any value $\eta < 1$, on the basis of these inequalities we will find that from the limitedness of particular solutions, whose existence is established by one of these theorems, there follows limitedness of the magnitude

$$\exp\int_{\tau}^{t} \mathrm{Re}\,\zeta_m\,dt$$

or magnitude

$$\exp\int_{\tau}^{t} \mathrm{Re}\,\lambda_m\,dt$$

correspondingly. But these magnitudes can be limited only in the case

$$\overline{\lim_{t\to\infty}}\int_{\tau}^{t} \mathrm{Re}\,\zeta_m\,dt < \infty$$

or

$$\overline{\lim_{t\to\infty}}\int_{\tau}^{t} \mathrm{Re}\,\lambda_m\,dt < \infty$$

correspondingly. From these inequalities, due to inequalities (5.14), there correspondingly follows

$$\overline{\lim_{t\to\infty}}\int_{\tau}^{t} \mathrm{Re}\,\zeta_i\,dt < \infty \qquad (i = 1, \ldots, n) \tag{5.[illegible]}$$

or

$$\overline{\lim_{t\to\infty}}\int_{\tau}^{t} \mathrm{Re}\,\lambda_i\,dt < \infty \qquad (i = 1, \ldots, n). \tag{5.[illegible]}$$

Hence, if there are executed conditions of the first or second theorem of the preceding paragraph and oscillations are stable, then during a certain sufficiently

large value of T there are executed inequalities (5.35). Analogously, fulfillment of conditions of the third theorem in the case of stable oscillations leads to inequalities (5.36).

Since, because of conditions (5.16) and (5.32), magnitudes Re ζ_m and Re λ_m during sufficiently large T do not change sign in the interval (T, ∞), then from inequalities (5.14) and (5.35) or (5.33) and (5.36) we will obtain

a) in cases Re $\zeta_m < 0$ or Re $\lambda_m < 0$ inequalities

$$\mathrm{Re}\,\zeta_i < 0 \text{ или } \mathrm{Re}\,\lambda_i < 0 \qquad (i = 1, \ldots, n);$$

b) in the case Re $\zeta_m > 0$ or Re $\lambda_m > 0$ inequalities

$$\overline{\lim_{t\to\infty}} \int_T^t \mathrm{Re}\,\zeta_m\,dt = \overline{\lim_{t\to\infty}} \int_T^t |\mathrm{Re}\,\zeta_m|\,dt < \infty$$

or

$$\overline{\lim_{t\to\infty}} \int_T^t \mathrm{Re}\,\lambda_m\,dt = \overline{\lim_{t\to\infty}} \int_T^t |\mathrm{Re}\,\lambda_m|\,dt < \infty,$$

i.e., conditions of absolute integrability of functions Re $\zeta_m(t)$ or Re $\lambda_m(t)$ in interval (T, ∞).

If oscillations are asymptotically stable and there are executed conditions of one of the theorems of the preceding section, then analogous reasonings lead to inequalities

$$\lim_{t\to\infty} \int_T^t \mathrm{Re}\,\zeta_i\,dt = -\infty \qquad (i = 1, \ldots, n) \tag{5.37}$$

or

$$\lim_{t\to\infty} \int_T^t \mathrm{Re}\,\lambda_i\,dt = -\infty \qquad (i = 1, \ldots, n). \tag{5.38}$$

In this case all magnitudes Re $\zeta_i(t)$ or Re $\lambda_i(t)$ must be negative in interval (T, ∞), where the possibility of absolute integrability of these functions in the shown interval is excluded.

Obtained results we will formulate in the form of the following theorem.

Theorem. In the conditions of one of the first or third theorems of the preceding paragraph, for stability of oscillations it is necessary that for any value of T there are executed inequalities (5.35) or (5.36) respectively; for asymptotic stability it is necessary that for any value of T there are executed inequalities (5.37) or (5.38), respectively.

Obviously, if the necessary condition of stability is disturbed, i.e., any inequality from systems of inequalities (5.35) or (5.36) is not executed, then oscillations are unstable; if the necessary condition of asymptotic stability is disturbed, i.e., there is not executed any inequality of systems of inequalities

(5.37) or (5.38), then oscillations are either unstable or stable but not asymptotic.

From conditions (5.37) or (5.38), because of conditions (5.14) or (5.33), there follows

$$\operatorname{Re}\zeta_i < 0 \quad (i = 1, \ldots, n) \quad (T < t < \infty) \tag{5.39}$$

or

$$\operatorname{Re}\lambda_i < 0 \quad (i = 1, \ldots, n) \quad (T < t < \infty). \tag{5.40}$$

Consequently, conditions (5.39) and (5.40) also are necessary conditions of asymptotic stability. They, in general, are less strong than conditions (5.37) or (5.38) [they allow the case of absolute integrability of functions $\zeta_i(t)$ or $\lambda_i(t)$], but are more convenient in application.

It is possible to check fulfillment of conditions (5.39) and (5.40) without calculation of roots $\zeta_i^{(l-1)}$ and λ_i ($i = 1, \ldots, n$). With this goal it is sufficient to determine whether coefficients of equations corresponding to these roots

$$(\zeta^{(l-1)})^n + b_1^{(l-1)}(\zeta^{(l-1)})^{n-1} + \ldots + b_n^{(l-1)} = 0$$

or

$$\lambda^n + b_1\lambda^{n-1} + \ldots + b_n = 0$$

satisfy conditions of Hurwitz or Routh.

Let us give the conditions of Routh.

Applying designations of the second equation, we will constitute matrix

$$\left.\begin{matrix} 1 & b_2 & b_4 \ldots \\ b_1 & b_3 & b_5 \ldots \\ c_0 & c_1 & c_2 \ldots \\ d_0 & d_1 & d_2 \ldots \\ \ldots & \ldots & \ldots \end{matrix}\right| ,$$

each line of which, starting from the third, is determined from the two preceding according to the following rule: from the elements of the upper line are subtracted corresponding elements of the lower, preliminarily multiplied by such a number that the first difference turns into zero. This zero difference is rejected and the obtained line is displaced by an element to the left (as compared to the corresponding elements of the upper lines) [33].

Conditions of Routh are connected with this diagram in the following manner: all elements of the first column of the diagram have to be different than zero and positive [33].

<u>Example</u>: In the preceding paragraph, for equation

$$\ddot{x} + ct^{\nu}x = 0, \quad c > 0, \quad \nu > -2$$

It was shown that for $\nu \neq 0$, during sufficiently large values of t, conditions (5.1[illegible]) and (5.16) are executed for system of equations relative to canonical components of the second expansion. Since

$$\operatorname{Re} \zeta_{1,2}^{(2)} = -\frac{\nu}{4}.$$

then conditions (5.39) of asymptotic stability are executed during $\nu > 0$. Consequently, for asymptotic stability of oscillations, it is necessary that there be executed inequality

$$\nu > 0.$$

If $\nu < 0$, then there is not executed only condition (5.39) but also condition (5.35). Consequently, at $\nu < 0$ oscillations are unstable.

C H A P T E R   VI

ASYMPTOTIC PRESENTATIONS OF SOLUTIONS OF AN EQUATION OF OSCILLATIONS

§ 1. Asymptotic Presentations of Particular Solutions of an Equation of Oscillations

Solution of the problem about stability of oscillations gives answers to the questions: are all particular solutions of an equation of free oscillations limited, and, if they are limited, are they vanishing (i.e., approaching zero at $t \to \infty$) functions of time? More complete judgement about asymptotic properties of free oscillations can be carried out if there are known asymptotic presentations of certain particular solutions of an equation of oscillations. After determining asymptotic presentations n of linearly independent solutions, there can be obtained also answers to questions about stability, without applying the theorems of the preceding paragraphs. Below will be considered questions connected with the idea and construction of asymptotic presentations of particular solutions of equations with p-multiple, differentiable coefficients.

The idea of an asymptotic presentation of a particular solution we will connect with a particular solution of an equation of oscillations, possessing certain definite properties, not being confused by the circumstance that it will not be applicable to all particular solutions. We will consider only such particular solutions for which in certain interval (t_0, ∞) is executed condition

$$x(t) \neq 0. \qquad (1.1)$$

To such solutions, in particular, pertain solutions satisfying the following conditions:

$$\left.\begin{array}{ll} \text{a)} & \underline{\lim}_{t\to\infty} \operatorname{Re} \operatorname{Ln} x(t) = \overline{\lim}_{t\to\infty} \operatorname{Re} \operatorname{Ln} x(t) = \pm\infty \\ \text{or} & \\ \text{b)} & -\infty < \underline{\lim}_{t\to\infty} \operatorname{Re} \operatorname{Ln} x(t) \leqslant \overline{\lim}_{t\to\infty} \operatorname{Re} \operatorname{Ln} x(t) < \infty; \\ \text{c)} & \underline{\lim}_{t\to\infty} \operatorname{Im} \operatorname{Ln} x(t) = \overline{\lim}_{t\to\infty} \operatorname{Im} \operatorname{Ln} x(t) = \pm\infty \\ \text{or} & \\ \text{d)} & -\infty < \underline{\lim}_{t\to\infty} \operatorname{Im} \operatorname{Ln} x(t) = \overline{\lim}_{t\to\infty} \operatorname{Im} \operatorname{Ln} x(t) < \infty. \end{array}\right\} \quad (6.2)$$

If particular solution x(t) of equation (0.1) satisfies conditions (6.1), then its asymptotic presentation we will call function X(t), connected with function x(t) by dependence

$$X(t) = x(t)\,v(t), \tag{6.3}$$

where v(t) is a function satisfying condition

$$v(t) \sim [x(t)]^{\operatorname{Re} o(1)} \quad \text{during} \quad t \to \infty. \tag{6.4}$$

Condition (6.4) one should understand so: it is possible to select such real function $\alpha(t) \to 0$ that there will be executed equality[1]

$$\lim_{t\to\infty} \frac{v(t)}{[x(t)]^{\alpha(t)}} = 1. \tag{6.5}$$

Function x^{α} during $\alpha < 1$ is many-valued in the region of complex values of argument x. Its many-valued nature follows from equality $x = x \exp 2k\pi i$ (k is any integer, $i = \sqrt{-1}$), because of which every value of x corresponds to finite or infinite set of values x^{α}, determined by equality

$$x^{\alpha} = \exp(\alpha \ln x) \exp 2k\pi\alpha i.$$

In this equality ln x is the principal value of logarithm [48], characterized by the fact that its imaginary component is contained in interval $(-\pi, \pi]$, i.e.,

$$-\pi < \operatorname{Im} \ln x \leqslant \pi.$$

Determining function v(t) by condition (6.4), we admit that there can be considered any branch of function $f(x) = x^{\alpha}$. It is assumed only that this function continuously depends on argument, i.e., does not jump from one branch to another.[2]

Let us consider certain results of condition (6.4).

Taking the logarithm of function v(t) and $[x(t)]^{\operatorname{Re} o(1)}$, because of this condition we will obtain

[1]Here and subsequently, during application of symbol lim, it is implied that (designated by it) a limit exists.

[2]In other words, it is assumed that magnitude x^2 continuously changes with change of t.

$$\operatorname{Ln} v(t) = \operatorname{Re} o(1) \operatorname{Ln} x(t) + 2k\pi i + o(1),$$

where $i = \sqrt{-1}$, k is an arbitrary integer, appearing because of the many-valued nature of the logarithmic function in the region of complex values of argument.[1]

Dividing real and imaginary parts, from this equality we will obtain the following system of two equalities

$$\left.\begin{aligned} \operatorname{Re} \operatorname{Ln} v(t) &= \operatorname{Re} o(1) \operatorname{Re} \operatorname{Ln} x(t) + \operatorname{Re} o(1), \\ \operatorname{Im} \operatorname{Ln} v(t) &= \operatorname{Re} o(1) \operatorname{Im} \operatorname{Ln} x(t) + 2\pi k + \operatorname{Re} o(1). \end{aligned}\right\}$$

Hence, in particular, there follows[2]

$$\left.\begin{aligned} &\lim_{t\to\infty} \frac{\operatorname{Re} \operatorname{Ln} v(t)}{\operatorname{Re} \operatorname{Ln} x(t)} = 0 \text{ — in case (6.2a)}, \\ &\lim_{t\to\infty} \operatorname{Re} \operatorname{Ln} v(t) = 0 \text{ — in case (6.2б)}, \\ &\lim_{t\to\infty} \frac{\operatorname{Im} \operatorname{Ln} v(t)}{\operatorname{Im} \operatorname{Ln} x(t)} = 0 \text{ — in case (6.2в)}, \\ &\lim_{t\to\infty} \operatorname{Im} \operatorname{Ln} v(t) = 2\pi k \text{ (where k is an integer — in case (6.[illegible])} \end{aligned}\right\} \quad (6.6)$$

If solution of equation of oscillations satisfies conditions (6.2), then condition (6.6) together with condition (6.3) can be taken for determination of asymptotic presentation. In this determination there are no values, whatever branches of logarithmic functions are considered. It is assumed only that they continuously depend on their arguments (i.e., do not jump from one branch to another).

Condition (6.4) is equivalent to condition

$$v(t) \sim [X(t)]^{\operatorname{Re} o(1)} \quad \text{for } t \to \infty. \quad (6.7)$$

Really, from conditions (6.3) and (6.5) follows

$$v \sim X^{\frac{\epsilon}{1-\epsilon}},$$

whence we obtain (6.7).

In particular, during fulfillment of conditions (6.2) we will obtain

---

[1]Many-valued nature of the logarithmic function in the region of complex values of argument w follows from equality

$$w = w \exp 2k\pi i$$

(where k is any integer), because of which every value of w corresponds to an infinite set of values Ln w, differing from each other by $2k\pi i$ (48).

[2]In the first two of the below-mentioned equalities and, subsequently, during designations of real parts of logarithms, symbol Ln u can be replaced by symbol ln u, since there occurs identity

$$\operatorname{Re} \operatorname{Ln} u = \operatorname{Re} \ln u.$$

$$\left.\begin{array}{ll}\lim\limits_{t\to\infty}\dfrac{\mathrm{Re\,Ln}\,v(t)}{\mathrm{Re\,Ln}\,X(t)}=0 - \text{in case} & (6.2a),\\ \lim\limits_{t\to\infty}\mathrm{Re\,Ln}\,v(t)=0 - \text{in case} & (6.2б),\\ \lim\limits_{t\to\infty}\dfrac{\mathrm{Im\,Ln}\,v(t)}{\mathrm{Im\,Ln}\,X(t)}=0 - \text{in case} & (6.2в),\\ \lim\limits_{t\to\infty}\mathrm{Im\,Ln}\,v(t)=2k\pi - \text{in case} & (6.2г).\end{array}\right| \qquad (6.8)$$

If there are known certain particular solutions of an equation of oscillations, satisfying conditions (6.1), then very simply may be constructed as large a number as desired of different asymptotic presentations of these solutions.

First, this solution itself is a trivial, asymptotic presentation of particular solution x(t), satisfying conditions (6.1). It is obtained from equality (6.3), if one were to assume $v(t) \equiv 1$; condition (6.4) for this form of function v(t) is always satisfied.

Secondly, it is possible to construct any amount many of its other asymptotic presentations, selecting function v(t), different from the above-mentioned, but satisfying conditions (6.4). Functions v(t) of such form for any solutions x(t) satisfying conditions (6.1) are, for instance, any functions approaching unity as $t \to \infty$.

The problem of constructing asymptotic presentations of particular solutions by given solutions has an infinite number of evident solutions but does not present interest. If certain particular solutions of equation of oscillations are known, then construction of their asymptotic presentations will not supplement our information about asymptotic properties of oscillations.

Of an entirely different character is the problem of constructing asymptotic presentations of certain particular solutions of an equation of oscillations in cases when these particular solutions themselves are unknown. In order to grasp how interesting this problem is, one should mention two circumstances:

a) asymptotic presentations of particular solutions characterize many asymptotic properties of free oscillations in the same measure as actual particular solutions (see §§ 3 and 4),

b) to determine asymptotic presentations of particular solutions in many cases is simpler than to find particular solutions.

Certainly, it is impossible to expect that solution of the problem of constructing asymptotic presentations of unknown particular solutions may be obtained by a simple and common method for various types of equations of free oscillations. This problem is very complicated, and the manner of its solution depends on the form of

coefficients of the equation of oscillations.

Determination of asymptotic presentations of unknown particular solutions is considered in the following paragraph.

<u>Example</u>: Equation

$$\ddot{x} - (t^2 + a)x = 0, \quad a > 0 \tag{6.9}$$

has the two following particular solutions:

$$x_1(t) = \exp\frac{t^2}{2}; \quad x_2(t) = \exp\frac{t^2}{2}\int \exp(-t^2)\,dt \tag{6.10}$$

(see § 7 Chapter III). Both these solutions satisfy conditions (6.2) where take place cases (a) and (d).

One of asymptotic presentations of the first solution is function

$$X_1(t) = t^{\alpha}\exp\frac{t^2}{2}, \tag{6.11}$$

where α is any real number.

Actually, here

$$r(t) = r_1(t) = t^{\alpha}.$$

This function satisfies conditions (6.6), since

$$\lim_{t\to\infty}\frac{\operatorname{Re}\operatorname{Ln} r_1(t)}{\operatorname{Re}\operatorname{Ln} x_1(t)} = \frac{2\alpha\ln t}{t^2} = 0,$$
$$\operatorname{Im}\operatorname{Ln} x_1(t) = 2\pi k_1, \quad \operatorname{Im}\operatorname{Ln} r_1(t) = 2\pi k_2$$

(k_1 and k_2 are any integers).

One of the asymptotic presentations of the second solution is function

$$X_2(t) = -t^{\alpha}\exp\left(-\frac{t^2}{2}\right), \tag{6.12}$$

since function

$$r_2(t) = \frac{-t^{\alpha}\exp(-t^2)}{\int \exp(-t^2)\,dt}$$

satisfies conditions (6.8).

Really,

$$\int \exp(-t^2)\,dt = \int\left(-\frac{1}{2t}\right)[-2t\exp(-t^2)\,dt] = -\frac{\exp(-t^2)}{2t} -$$
$$-\int \exp(-t^2)\frac{1}{2t^2}\,dt = -\frac{\exp(-t^2)}{2t}[1 + |o(1)|],$$
$$\operatorname{Ln} r_2(t) = (\alpha + 1)\ln t + \ln 2 - \ln[1 + |o(1)|] + 2k_2\pi i,$$
$$\lim_{t\to\infty}\frac{\operatorname{Re}\operatorname{Ln} r_2(t)}{\operatorname{Re}\operatorname{Ln} X_2(t)} = \lim_{t\to\infty}\frac{(\alpha + 1)\ln t}{\alpha\ln t - \frac{t^2}{2}} = 0,$$
$$\operatorname{Im}\operatorname{Ln} X_2(t) = (2k_1 + 1)\pi,$$
$$\operatorname{Im}\operatorname{Ln} r_2(t) = 2k_2\pi$$

(k_1 and k_2 are any integers).

§ 2. Methods of Determining Asymptotic Presentations of Particular Solutions

The simplest cases for construction of asymptotic presentations of unknown particular solutions are those in which the matrices of coefficients of the system of equations relative to canonical components (2.48) possess the following properties:

a) the matrix is L-diagonal [38], i.e., such that all coefficients $h_{ij}^{(k)}$ during certain $t_0 > 0$ are absolutely integrable[1] in interval (t_0, ∞);

b) there exists such sufficiently large T, that during $t \geq T$ not one of differences $\operatorname{Re} \zeta_i^{(k-1)}(t) - \operatorname{Re} \zeta_j^{(k-1)}(t)$ changes sign.

For analysis of these cases, the below-mentioned theorem of I. M. Rapoport [38] is applicable.

Theorem of Rapoport. If the matrix of coefficients of a system of equations, recorded in the form (2.48) possesses properties (a) and (b), and functions $\zeta_i^{(k-1)}(t)$ $(i = 1, \ldots, n)$ are absolutely integrable in interval $t_0, t_1)$ during any finite t_1, then this system of equations has n particular solutions of the form

$$z_i = \eta_{ij}(t) \exp \int_{t_0}^{t} \zeta_j^{(k-1)}(t)\, dt \qquad (i = 1, 2, \ldots, n), \tag{6.13}$$

where $\eta_{ij}(t)$ are functions, continuous in interval $[t_0, \infty)$, with which $\eta_{ii}(\infty) = 1$ and $\eta_{ij}(\infty) = 0$ during $i \neq j$.

Assuming that conditions of Rapoport's theorem are executed, and using formula (6.13) and property of canonical components

$$\sum_{i=1}^{n} z_i = x,$$

we will obtain n particular solutions of equation (0.1) in the form

$$x_j = \left[\sum_{i=1}^{n} \eta_{ij}(t)\right] \exp \int_{t_0}^{t} \zeta_j^{(k-1)}(t)\, dt, \tag{6.14}$$

where

$$\lim_{t\to\infty} \sum_{i=1}^{n} \eta_{ij}(t) = 1.$$

Since magnitudes

$$\exp \int_{t_0}^{t} \zeta_j^{(k-1)}(t)\, dt \qquad (j = 1, \ldots, n)$$

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[1] Function $h(t)$ is called absolutely integrable in interval (a, b) if integral

$$\int_a^b |h(t)|\, dt$$

converges (48).

do not turn into zero during any value of t from interval (t_0, ∞) solutions (6.14) satisfy condition (6.1). Because of shown properties of function $\eta_{ij}(t)$ one of the asymptotic presentations of each j-th solution is the following

$$X_j = \exp\int_{t_0}^{t} \zeta_j^{(k-1)}(t)\,dt. \tag{6.15}$$

Let us note that if magnitude $\zeta_j^{(k-1)}$ is real, then among solutions with asymptotic presentation (6.15) there must be a real resolution, but if magnitudes $\zeta_j^{(k-1)}$ and $\zeta_{j+m}^{(k-1)}$ are conjugate complex, then among solutions with asymptotic presentations

$$X_j = \exp\int_{t_0}^{t} \zeta_j^{(k-1)}(t)\,dt \text{ and } X_{j+m} = \exp\int_{t_0}^{t} \zeta_{j+m}^{(k-1)}(t)\,dt$$

there must be a pair of solutions taking conjugate complex values.

Actually, if magnitude $\zeta_j^{(k-1)}(t)$ is real, then magnitude η_{jj} also may be real. Considering differentiability of functions $\eta_{ij}(t)$ [following from formulas (6.13)] and passing in system of equations (2.48) from variables z_i ($i = 1, \ldots, n$) to variables η_{ij} ($i = 1, \ldots, n$) we will obtain system of equations [38]

$$\dot{\eta}_{ij} = [\zeta_i^{(k-1)}(t) - \zeta_j^{(k-1)}(t)]\,\eta_{ij} + \sum_{l=1}^{n} h_{il}^{(k)}(t)\,\eta_{lj} \tag{6.16}$$

$$(i = 1, \ldots, n).$$

Solutions of this system, corresponding to real solutions of equation (6.1), possess the property that during real functions $\zeta_j^{(k-1)}(t)$ complex-valued functions $\eta_{ij}(i \neq j)$ are united into conjugate pairs and, consequently, sum

$$\sum_{i=1}^{n} \eta_{ij}(t)$$

is real.[1]

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[1]There is possible other proof. Let us assume that functions $\eta_{ij}(t)$ interesting us, are complex-valued. Then, in right sides of formulas (6.13) to complex conjugate magnitudes, we will obtain one more particular solution of equation (6.1), consisting of sum of functions of the form

$$z_i = \bar{\eta}_{ij}(t)\exp\int_{t_0}^{t} \zeta_j^{(k-1)}(t)\,dt$$

$$(i = 1, \ldots, n).$$

Obviously, sum of functions

$$z_i = \frac{\eta_{ij}(t) + \bar{\eta}_{ij}(t)}{2}\exp\int_{t_0}^{t} \zeta_j^{(k-1)}(t)\,dt \quad (i = 1, \ldots, n)$$

also is particular solution of an equation of oscillations. The validity of the expressed statement follows from this.

If magnitudes $\zeta_j^{(k-1)}(t)$ and $\zeta_{j+m}^{(k-1)}(t)$ are conjugate complex, then, considering system (6.16), it is possible to establish the existence of two of its solutions, for which magnitudes

$$\sum_{i=1}^{n} \tau_{ij} \text{ and } \sum_{i=1}^{n} \tau_{i,j+m} \text{ — are also conjugate complex}$$

Thereby the expressed-above affirmation is proven.

The fixed result we will formulate in the form of the following theorem.

Theorem. If the matrix of coefficients of system (2.48) satisfies the condition of Rapoports theorem, then equation (0.1) has n particular solutions $x_j(t)$, each of which has an asymptotic presentation of the form (6.15); among the solutions corresponding to real asymptotic presentations is a real solution, and among the solutions corresponding to complex conjugate asymptotic presentations are complex conjugate solutions.

Let us assume that an asymptotic presentation of the form (6.15) of certain particular solution $x_j(t)$ is found. Then with the help of functions $\gamma_j(t)$, satisfying condition

$$\left.\begin{aligned} \int_{t_0}^{t} \operatorname{Re}\gamma_j(t)\,dt &= \operatorname{Re} o(1) \int_{t_0}^{t} \operatorname{Re}\zeta_j^{(k-1)}(t)\,dt + \operatorname{Re} o(1), \\ \int_{t_0}^{t} \operatorname{Im}\gamma_j(t)\,dt &= \operatorname{Re} o(1) \int_{t_0}^{t} \operatorname{Im}\zeta_j^{(k-1)}(t)\,dt + 2\pi k + \operatorname{Re} o(1), \end{aligned}\right\} \tag{6.17}$$

it is possible to construct an infinite set of other asymptotic presentations of the form

$$X_j = \exp\int_{t_0}^{t} [\zeta_j^{(k-1)}(t) + \gamma_j(t)]\,dt. \tag{6.18}$$

Let us assume now that $\gamma_j(t)$ are arbitrary functions satisfying conditions (6.17), and they are established in the form (6.18) m ($m \leq n$) of asymptotic presentations

$$X_1(t), \ldots, X_m(t)$$

of certain particular solutions. It is easy to see that particular solutions corresponding to them

$$x_1(t), \ldots, x_m(t)$$

are linearly independent if there does not exist such functions $\gamma_j(t)$ ($j = 1, \ldots, m$) satisfying condition (6.17), at which these asymptotic presentations are linearly dependent.

Actually, if we assume that particular solutions are linearly dependent, then, after selection function $\gamma_j(t)$ ($j = 1, \ldots, m$) in such a manner that asymptotic

presentations of solutions coincide with solutions, we will find a system of asymptotic presentations connected by linear dependence, and, consequently, we will come to a contradiction.

The established conformity between properties of particular solutions and their asymptotic presentations carried the character of a sufficient condition of linear independence of particular solutions given by their asymptotic presentations.

Thus, we have established a method of constructing asymptotic presentations of particular solutions and conditions which asymptotic presentations must satisfy so that particular solutions are linearly independent in cases when the matrix of coefficients of a system of equations relative to canonical components possess the above-indicated properties (a) and (b). It is possible to see the presence or absence for the mentioned matrix of property (a) if one were to calculate coefficients $h_{ij}^{(k)}$ $(i, j = 1, \ldots, n)$ and check their integrability. However, calculation of coefficients is a very labor-consuming operation since it is connected with differentiation and algebraic transformations of roots of algebraic equation $\zeta_1^{(k-1)}, \ldots, \zeta_n^{(k-1)}$ [see formula for coefficient $h_{ij}^{(k+1)}$ in explanation to equation (2.49)]. Therefore, of interest are methods of checking integrability of coefficients $h_{ij}^{(k)}$ $(i, j = 1, \ldots, n)$, which do not require their calculation.

Such methods are possible. One of them, which we will consider, is based on the following lemma.

<u>Lemma</u>. <u>The necessary and sufficient condition of absolute integrability of functions</u> $\varkappa_1(t), \ldots, \varkappa_n(t)$ <u>in interval</u> (t_0, ω) <u>is the fulfillment of inequalities</u>

$$\int_{t_0}^{\omega} |k_i(t)|^{1/i}\, dt < \infty \qquad (i = 1, \ldots, n), \tag{2.[illegible]}$$

where

$$k_1(t) = \sum_{i=1}^{n} \varkappa_i(t);$$

$$k_2(t) = \sum_{\substack{i,j=1 \\ j>i}}^{n} \varkappa_i(t)\varkappa_j(t);$$

$$\cdots\cdots\cdots\cdots$$

$$k_n(t) = \varkappa_1(t) \ldots \varkappa_n(t).$$

<u>Proof</u>. Let us agree to use the term "integrability" subsequently in the sense of absolute integrability.

If functions $\varkappa_1(t), \ldots, \varkappa_n(t)$ are integrable in interval (t_0, ω), then their sum — function $k_1(t)$ — also is integrable in interval (t_0, ω). Further for arbitrary function $k_m(t)$ $(m = 2, \ldots, n)$

$$\int_{t_0}^{\infty} |k_m(t)|^{1/m} dt = \int_{t_0}^{\infty} \Big| \sum_{i_j \neq i_k} \varkappa_{i_1}(t)\varkappa_{i_2}(t)\dots\varkappa_{i_m}(t)\Big|^{1/m} dt \leq$$

$$\leq \int_{t_0}^{\infty}\Big(\sum_{i_j\neq i_k} |\varkappa_{i_1}(t)\varkappa_{i_2}(t)\dots\varkappa_{i_m}(t)|\Big)^{1/m} dt \leq$$

$$\leq \int_{t_0}^{\infty} \sum_{i_j\neq i_k} |\varkappa_{i_1}(t)\varkappa_{i_2}(t)\dots\varkappa_{i_m}(t)|^{1/m} dt$$

(in the last relationship we can be convinced, raising the integrand to m power). On the basis of inequality of Gelder[1]

$$\int_{t_0}^{\infty} \sum_{i_j\neq i_k} |\varkappa_{i_1}(t)\varkappa_{i_2}(t)\dots\varkappa_{i_m}(t)|^{1/m} dt \leq$$

$$\leq \sum_{i_j\neq i_k}\Big(\int_{t_0}^{\infty}|\varkappa_{i_1}|dt\Big)^{1/m}\Big(\int_{t_0}^{\infty}|\varkappa_{i_2}|dt\Big)^{1/m}\dots\Big(\int_{t_0}^{\infty}|\varkappa_{i_m}|dt\Big)^{1/m}.$$

But since functions $\varkappa_{i_j}(t)$ are integrable, the right side of the last inequality is limited. Consequently,

$$\int_{t_0}^{\infty} |k_m(t)|^{1/m} dt < \infty.$$

Thus, the <u>necessity</u> of condition (6.19) is proven.

For proof of sufficiency of condition (6.19) we will assume that it is carried out, and we will observe whether there can be cases when functions $\varkappa_i(t)$ (one or several) are not integrable.

Let us assume that only one of functions $\varkappa_i(t)$ is not integrable. Then because of equality

$$k_1(t) = \sum_{n} \varkappa_i(t) \tag{6.20}$$

this function it is possible to present in the form of a sum of integrands, which, obviously, is also integrable, and, thus, we arrive at a contradiction.

Let us assume that there are only two functions which are not integrable, for instance $\varkappa_1(t)$ and $\varkappa_2(t)$. Then because of relationship (6.19), sum of functions $\varkappa_1(t) + \varkappa_2(t)$ is integrable, as a consequence of which it is possible to record

$$\varkappa_2(t) = -\varkappa_1(t) + \nu_1(t). \tag{6.21}$$

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[1] Inequality of Gelder has the form [34]

$$\int f^{\alpha} g^{\beta} \dots l^{\lambda} dt \leq \Big(\int f\,dt\Big)^{\alpha}\Big(\int g\,dt\Big)^{\beta}\dots\Big(\int l\,dt\Big)^{\lambda},$$

where

$$f = f(t) > 0,\ g = g(t) > 0, \dots,\ l = l(t) > 0,$$

$\alpha, \beta, \dots, \lambda$ are positive numbers where $\alpha + \beta + \dots + \lambda = 1$.

where $\nu_1(t)$ is the integrand.

After carrying out substitution (6.21) in equality

$$k_2(t) = \sum_{i<j} x_i(t) x_j(t),$$

we will obtain

$$k_2(t) = -x_1(t)[x_1(t) - \nu_1(t)] + \nu_2(t). \tag{6.22}$$

where $\nu_2(t)$ is the sum of functions, each of which is the product of two integrands.

Because of equality (6.22)

$$x_1(t)[x_1(t) - \nu_1(t)] = \nu_2(t) - [k_2(t)]^{1/2}[k_2(t)]^{1/2} = \nu_2'(t). \tag{6.23}$$

where $\nu_2'(t)$ also is sum of functions, each of which is the product of two integrands.

From equality (6.23) follows

$$\sqrt{|[x_1(t)]^2 - x_1(t)\nu_1(t)|} = \sqrt{|\nu_2'(t)|}.$$

whence, in turn,

$$\int_{t_0}^{t} \sqrt{|[x_1(t)]^2 - x_1(t)\nu_1(t)|}\,dt = \int_{t_0}^{t} \sqrt{|\nu_2'(t)|}\,dt. \tag{6.24}$$

We will estimate left and right sides of equality (6.24). Because of evident algebraic inequality

$$\int_{t_0}^{t} \sqrt{|[x_1(t)]^2 - x_1(t)\nu(t)|}\,dt \geqslant \int_{t_0}^{t} |x_1(t)|\,dt - \int_{t_0}^{t} \sqrt{|x_1(t)\nu(t)|}\,dt.$$

On the basis of Buniakowski inequality

$$\int_{t_0}^{t} \sqrt{|x_1(t)\nu_1(t)|}\,dt \leqslant \sqrt{\int_{t_0}^{t} |x_1(t)|\,dt}\,\sqrt{\int_{t_0}^{t} |\nu_1(t)|\,dt}.$$

Consequently,

$$\begin{aligned}\int_{t_0}^{t} \sqrt{|[x_1(t)]^2 - x_1(t)\nu_1(t)|}\,dt &\geqslant \int_{t_0}^{t} |x_1(t)|\,dt - \\ &-\sqrt{\int_{t_0}^{t} |x_1(t)|\,dt}\,\sqrt{\int_{t_0}^{t} |\nu_1(t)|\,dt} = \\ &= \sqrt{\int_{t_0}^{t} |x_1(t)|\,dt}\left(\sqrt{\int_{t_0}^{t} |x_1(t)|\,dt} - \sqrt{\int_{t_0}^{t} |\nu_1(t)|\,dt}\right).\end{aligned} \tag{6.25}$$

Taking into account that

$$\int_{t_0}^{\infty} |x_1(t)|\,dt = \infty, \quad \int_{t_0}^{\infty} |\nu_1(t)|\,dt < \infty,$$

we will find that during $t \to \infty$ right side of inequality (6.25) without limit increases. Consequently, without limit increases also left part of this inequality, which is simultaneously left part of equality (6.24).

Applying Buniakowski inequality, it is easy to establish that the right side of equality (6.24) is a bounded function.

Comparing obtained results, we arrive at a contradiction. Consequently, the case of two nonintegrands is excluded.

The same diagram of construction is applicable for proof of the contradictory nature of the assumption about the existence m ($2 < m \leq n$) of nonintegrands among functions $\varkappa_1(t)$.

Let us assume that $\varkappa_1(t)$, $\varkappa_2(t)$, ..., $\varkappa_m(t)$ are nonintegrands. Then, using property of functions $k_1(t)$, ..., $k_m(t)$, given in conditions of lemma, it is possible to arrive at equality

$$x_1^m + \nu_1 x_1^{m-1} + \ldots + \nu_{m-1} x_1 = \nu_m, \tag{6.26}$$

where ν_i ($i = 1, \ldots, m$) is a certain sum of functions, each of which is the product of i integrands.

From equality (6.26) follows

$$|x_1^m + \nu_1 x_1^{m-1} + \ldots + \nu_{m-1} x_1|^{1/m} = |\nu_m|^{1/m},$$

whence, in turn,

$$\int_{t_0}^{t} |x_1^m + \nu_1 x_1^{m-1} + \ldots + \nu_{m-1} x_1|^{1/m}\, dt = \int_{t_0}^{t} |\nu_m|^{1/m} dt. \tag{6.27}$$

Because of known algebraic inequalities

$$\begin{gathered}\int_{t_0}^{t} |x_1^m + \nu_1 x_1^{m-1} + \ldots + \nu_{m-1} x_1|^{1/m}\, dt \geqslant \int_{t_0}^{t} |x_1|\, dt - \\ - \int_{t_0}^{t} |\nu_1 x_1^{m-1} + \ldots + \nu_{m-1} x_1|^{1/m}\, dt \geqslant \int_{t_0}^{t} |x_1|\, dt - \\ - \int_{t_0}^{t} |\nu_1 x_1^{m-1}|^{1/m}\, dt - \int_{t_0}^{t} |\nu_2 x_1^{m-2}|^{1/m}\, dt - \ldots - \int_{t_0}^{t} |\nu_{m-1} x_1|^{1/m}\, dt.\end{gathered} \tag{6.28}$$

On the basis of inequality of Gelder

$$\int_{t_0}^{t} |\nu_{m-i} x_1^i|^{1/m}\, dt \leqslant \left(\int_{t_0}^{t} |x_1|\, dt\right)^{i/m} \left(\int_{t_0}^{t} |\nu_{m-i}|^{\frac{1}{m-i}}\, dt\right)^{\frac{m-i}{m}} \quad (i = 1, 2, \ldots, m-1). \tag{6.29}$$

Comparing inequalities (6.28) and (6.29), we will obtain

$$\begin{gathered}\int_{t_0}^{t} |x_1^m + \nu_1 x_1^{m-1} + \ldots + \nu_{m-1} x_1|^{1/m}\, dt \geqslant \left(\int_{t_0}^{t} |x_1|\, dt\right)^{1/m} \left[\left(\int_{t_0}^{t} |x_1|\, dt\right)^{\frac{m-1}{m}} - \left(\int_{t_0}^{t} |x_1|\, dt\right)^{\frac{m-2}{m}} \left(\int_{t_0}^{t} |\nu_1|\, dt\right)^{1/m} - \ldots \right. \\ \left. \ldots - \left(\int_{t_0}^{t} |\nu_{m-1}|^{\frac{1}{m-1}}\, dt\right)^{\frac{m-1}{m}}\right].\end{gathered} \tag{6.30}$$

Taking into account that

$$\int_{t_0}^{\infty} |x_1(t)|\,dt = \infty, \quad \int_{t_0}^{\infty} |\nu_i(t)|^{1/m}\,dt < \infty$$
$$(i = 1, \ldots, m-1),$$

we will find that during $t \to \infty$ any difference

$$\frac{1}{m-1}\left(\int_{t_0}^{t}|x_1|\,dt\right)^{\frac{m-1}{m}} - \left(\int_{t_0}^{t}|x_1|\,dt\right)^{\frac{m-i-1}{m}}\left(\int_{t_0}^{t}|\nu_i|\,dt\right)^{i/m} =$$

$$= \left(\int_{t_0}^{t}|x_1|\,dt\right)^{\frac{m-i-1}{m}}\left[\frac{1}{m-1}\left(\int_{t_0}^{t}|x_1|\,dt\right)^{i/m} - \left(\int_{t_0}^{t}|\nu_i|^{1/i}\,dt\right)^{i/m}\right]$$
$$(i = 1, \ldots, m-1)$$

without limit increases. Consequently, without limit increases also second factor of right side of inequality (6.30). Obviously, the same property is possessed by the right side of inequality (6.30) and, consequently, by the left part of equality (6.27).

On the other hand, the right side of equality (6.27) is a bounded function.

Comparing obtained results, we arrive at a contradiction.

Thus, the lemma is proven.

We will apply this lemma to a check of integrability of coefficients $h_{ij}^{(k)}$ $(i, j = 1, \ldots, n)$.

With this goal, after designating the shown coefficients by symbols $\varkappa_1, \varkappa_2, \ldots, \varkappa_n$ and revealing their meaning by formulas given in explanations to formula (2.49), we will constitute an expression for coefficients $k_1, k_2, \ldots, k_n$. These expressions have the form of symmetric functions of roots $\zeta_1^{(k-1)}, \zeta_2^{(k-1)}, \ldots, \zeta_n^{(k-1)}$ and their derivatives. Therefore, they can be presented in the form of algebraic functions of coefficients $b_1^{(k-1)}, b_2^{(k-1)}, \ldots, b_n^{(k-1)}$ and their derivatives (and, of course, of coefficients $b_1, b_2, \ldots, b_n$), and consequently, also in the form of algebraic functions $b_1, b_2, \ldots, b_n$. Thus, functions $k_1(t), k_2(t), \ldots, k_n(t)$ can be constructed with respect to functions $b_1^{(k-1)}(t), b_2^{(k-1)}(t), \ldots, b_n^{(k-1)}(t)$ and $b_1(t), b_2(t), \ldots, b_n(t)$ by means of the application of only elementary algebraic operations and differentiation. After determining these functions check of integrability of coefficients $h_{ij}^{(k)}(t)$ on the basis of proven lemma can be replaced by check of fulfillment of conditions (6.19).

Thus, for instance, for a second order equation

$$k_1(t)=0,$$

$$k_2(t)=-\frac{[\dot{\zeta}_1^{(k-1)}+(\zeta_1^{(k-1)})^2+b_1(\zeta_1+\zeta_2)+\dot{\zeta}_2^{(k-1)}+(\zeta_2^{(k-1)})^2+2b_2]^2}{(\zeta_2^{(k-1)}-\zeta_1^{(k-1)})^2}+$$

$$+\frac{2[\dot{\zeta}_1^{(k-1)}+(\zeta_1^{(k-1)})^2+b_1\zeta_1^{(k-1)}+b_2][\dot{\zeta}_2^{(k-1)}+(\zeta_2^{(k-1)})^2+b_1\zeta_2^{(k-1)}+b_2]}{(\zeta_2^{(k-1)}-\zeta_1^{(k-1)})^2},$$

$$k_3(t)=0,$$

$$k_4(t)=\frac{[\dot{\zeta}_1^{(k-1)}+(\zeta_1^{(k-1)})^2+b_1\zeta_1^{(k-1)}+b_2]^2[\dot{\zeta}_2^{(k-1)}+(\zeta_2^{(k-1)})^2+b_1\zeta_2^{(k-1)}+b_2]^2}{(\zeta_2^{(k-1)}-\zeta_1^{(k-1)})^4}$$

Since magnitude $k_4(t)$ is equal to the square of half of the second component of magnitude $k_2(t)$, conditions (6.19) will take the form

$$\int_{t_0}^{\infty}\left|\frac{\dot{\zeta}_1^{(k-1)}+(\zeta_1^{(k-1)})^2+\dot{\zeta}_2^{(k-1)}+(\zeta_2^{(k-1)})^2+b_1(\zeta_1+\zeta_2)+2b_2}{\zeta_2-\zeta_1}\right|dt<\infty,$$

$$\int_{t_0}^{\infty}\sqrt{\left|\frac{[\dot{\zeta}_1^{(k-1)}+(\zeta_1^{(k-1)})^2+b_1\zeta_1^{(k-1)}+b_2][\dot{\zeta}_2^{(k-1)}+(\zeta_2^{(k-1)})^2+b_1\zeta_2^{(k-1)}+b_2]}{(\zeta_2-\zeta_1)^2}\right|}\,dt<\infty.$$

$$(6.31)$$

Expressing symmetric functions of roots and their derivatives through coefficients and their derivatives, we will give conditions (6.31) the form

$$\left.\begin{aligned}
&\int_{t_0}^{\infty}\sqrt{\left|\frac{b_1^{(k-1)}(\dot{b}_1^{(k-1)}-b_1)+2(b_2-b_2^{(k-1)})-\dot{b}_1^{(k-1)}}{(b_1^{(k-1)})^2-4b_2^{(k-1)}}\right|}\,dt<\infty,\\
&\int_{t_0}^{\infty}\sqrt{\left|\frac{A}{(b_1^{(k-1)})^2-4b_2^{(k-1)}}+\right.}\\
&\qquad\overline{+\frac{b_2^2-b_1b_2b_1^{(k-1)}+(b_2^{(k-1)})^2-b_1^{(k-1)}\dot{b}_2^{(k-1)}+\dot{b}_1^{(k-1)}b_2^{(k-1)}}{(b_1^{(k-1)})^2-4b_2^{(k-1)}}-}\\
&\qquad\overline{\left.-\frac{(\dot{b}_1^{(k-1)})^2b_2^{(k-1)}-b_1^{(k-1)}\dot{b}_1^{(k-1)}\dot{b}_2^{(k-1)}+(\dot{b}_2^{(k-1)})^2}{[(b_1^{(k-1)})^2-4b_2^{(k-1)}]^2}\right|}\,dt<\infty,\\
&\text{where } A=b_1(\dot{b}_2^{(k-1)}-b_1^{(k-1)}b_2^{(k-1)})+b_2[(b_1^{(k-1)})^2-\dot{b}_1^{(k-1)}-\\
&-2b_2^{(k-1)}]+b_1^2b_2^{(k-1)}.
\end{aligned}\right|\qquad(6.32)$$

More complicated cases for construction of asymptotic presentations of particular solutions are the cases in which canonical expansion does not lead to such a system of equations relative to canonical components, whose matrix of coefficients possesses properties (a) and (b) characteristic for the considered case. If the mentioned matrix does not possess property (a) but satisfies condition (b), then there can appear the useful method of auxiliary canonical expansions, presented below.

Let us assume that $\tilde{x}_j(t) = \exp \int \zeta_j^{(k-1)}(t)dt$ is an approximate presentation of a certain particular solution of equation (0.1), found by the method shown in § 2, Chapter IV by means of approximate solution of a system of equations relative to canonical components, obtained as a result of k-th canonical expansion of solution of equation (0.1). Then it satisfies equation (4.28)

$$\frac{d^n\tilde{x}}{dt^n} + b_1 \frac{d^{n-1}\tilde{x}}{dt^{n-1}} + \ldots + (b_n + \vartheta_j)\tilde{x} = 0,$$

which differs from equation (0.1) only by coefficient during unknown variable. Being limited by the case when

$$\lim_{t\to\infty} \frac{\vartheta_j}{b_n} = 0, \qquad (6.33)$$

we will replace equation (0.1) by equation

$$\frac{d^n x'}{dt^n} + b_{11} \frac{d^{n-1}x'}{dt^{n-1}} + \ldots + b_{n1}x' = 0, \qquad (6.34)$$

where

$$b_{k1} = b_k \qquad (k = 1, \ldots, n-1);$$
$$b_{n1} = b_n - \vartheta_j.$$

After carrying out k-th unmodulated canonical expansion of solution of equation (6.34) (we assume that conditions of applicability of this canonical expansion are executed), its approximate solution $\tilde{x}'_j$ we will find by the formula

$$\tilde{x}'_j = \exp \int \zeta_{j1}^{(k-1)}(t)\,dt, \qquad (6.35)$$

where $\zeta_{j1}^{(k-1)}$ is the root of algebraic equation

$$(\zeta^{(k-1)})^n + b_{11}^{(k-1)}(\zeta^{(k-1)})^{n-1} + \ldots + b_{n1}^{(k-1)} = 0, \qquad (6.36)$$

determining k-th expansion of solution of equation (6.34), corresponding to root $\zeta_j^{(k)}$ of equation

$$(\zeta^{(k-1)})^n + b_1^{(k-1)}(\zeta^{(k-1)})^{n-1} + \ldots + b_n^{(k-1)} = 0. \qquad (6.37)$$

The idea of conformity of roots is used here in the following meaning. If in equation (6.34) we replace coefficient b_{n1} by coefficient $b'_{n1} = b_n - m\vartheta_j$ and trace change of roots of equation during change of m from 0 to 1, then $\zeta_{j1}^{(k-1)}$ is final form of that root whose initial form is $\zeta_j^{(k-1)}$.

Coefficients of equation (6.36) are rationally expressed through coefficients and derivatives of coefficients of equation (6.34) and can be represented in the form

$$b_{i1}^{(k)} = b_i^{(k)} + \Delta b_i^{(k)} \qquad (i = 1, \ldots, n).$$

where $\Delta h_i^{(k)}(t)$ are functions depending on coefficients $b_1, \dots, b_n$ and their derivatives and on function $\vartheta_j(t)$ and its derivatives.

We will replace, in k-th canonical expansion of solution of equation (0.1), functions $\zeta_i^{(k-1)}(t)$ by functions $\zeta_{i1}^{(k-1)}(t)$. Then, system of equations relative to new canonical components will take the form

$$\dot{z}_i = \zeta_{i1}^{(k-1)} z_i + \sum_{j=1}^{n} h_{ij1}^{(k)} z_j \qquad (i = 1, \dots, n). \tag{6.38}$$

where $h_{ij1}^{(k)}$ are certain functions, in general, differing from functions $h_{ij}^{(k)}(t)$.

It is possible to expect that matrix of coefficients of system (6.38) possesses above-indicated properties (a) and (b). The basis to this is the following.

The equation which function $\tilde{x}'(t)$ satisfies it is possible to record in the form

$$\frac{d^n \tilde{x}'}{dt^n} + b_{11} \frac{d^{n-1} \tilde{x}'}{dt^{n-1}} + \dots + (b_{n1} + \vartheta_j') \tilde{x}' = 0.$$

where function ϑ_j' is found with respect to coefficients of equation (6.34) and root $\zeta_{j1}^{(k-1)}$ of equation (6.36) by the same method by which was determined function $\vartheta_j(t)$ with respect to coefficients of equation (0.1) and root $\zeta_j^{(k-1)}$ of equation (6.37). Since during $t \to \infty$

$$b_{n1} \sim b_n,$$

it is fully probable that

$$\zeta_{j1}^{(k-1)} \sim \zeta_j^{(k-1)} \qquad (j = 1, \dots, n) \text{ and } \vartheta_j' \sim \vartheta_j.$$

Because of the first relationships it is possible to expect that property (b) is possessed also by functions $\zeta_{j1}^{(k-1)}$ $(i = 1, \dots, n)$. On the basis of the last relationship and equality

$$b_{n1} + \vartheta_j' = b_n - \vartheta_j + \vartheta_j'$$

it is possible to expect that during $t \to \infty$ the order of proximity of coefficients $b_{n1} + \vartheta_r'$ and b_n is higher than the order of proximity of coefficients $b_n + \vartheta_j$ and $b_n \left(\lim_{t \to \infty} \frac{b_{n1} + \vartheta_j' - b_n}{\vartheta_j} = 0 \right)$, approximate solution $\tilde{x}_j'(t)$ is nearer to exact than approximate solution $\tilde{x}_j(t)$, and matrix of coefficients of system (6.38) is nearer to diagonal than matrix of coefficients of system (2.48). From the last, follows the possibility of fulfillment of condition (a).

If, as a result of check, there will be revealed properties (a) and (b) for matrix of coefficients of system (6.38), then it may be concluded that equation

(0.1) has particular solution, one of the asymptotic presentations of which has the form

$$\tilde{x}_j = \exp\int \zeta_{ji}^{(k-1)}(t)\,dt.$$

This method of constructing asymptotic presentations of particular solutions can be very labor-consuming. The following small modification of it will allow us, in a number of cases, essentially to reduce the volume of calculating operations: in equation (6.34) during determination of coefficient b_{n1} we will replace equality $b_{n1} = b_1 - \vartheta_j$ by equality $b_{n1} = b_n - \vartheta_j + o(\vartheta_j)$ and will select magnitude $o(\vartheta_j)$ in such a way that expression for coefficient $b_n(t)$ is the most convenient for accomplishing the foreseen (by the method) operation on function $b_{n1}(t)$. Such deviation from the presented method will not be reflected on above-mentioned reasonings connected with its logical foundation.

Finishing the account of methods of constructing asymptotic presentations of particular solutions, let us note that in the simplest cases, in which the matrix of coefficients of a system of equations relative to canonical components in l-th expansion is diagonal, and, consequently, asymptotic presentations can be obtained by means of integrating the mentioned system, between coefficients $b_j^{(l)}$ and $b_j^{(l-1)}$ $(j = 1, \ldots, n)$ take place dependences $b_j^{(l)} = b_j^{(l-1)}$ $(j = 1, \ldots, n)$ (see § 4, Chapter II). Because of this, in the above-considered, more general cases, as the initial number for constructing asymptotic presentations, it is expedient to take that number of canonical expansions of k, at which between coefficients $b_j^{(k)}$ and $b_j^{(k-1)}$ $(j = 1, \ldots, n)$ take place dependences

$$b_j^{(k)} = b_j^{(k-1)} + o(b_j^{(k-1)}) \quad (j = 1, \ldots, n). \tag{6.39}$$

However, one should consider that we do not have bases to negate the possibility of determining asymptotic presentations even with such numbers of canonical expansions at which condition (6.39) is not executed.

Example: In the case of equation (4.41)

$$\ddot{x} + ct^{\nu}x = 0, \quad c \neq 0, \quad \nu > -2$$

(see example in § 2 of the preceding chapter) canonical expansions give

$$b_1^{(1)} = -\frac{\nu}{2t}; \quad b_2^{(1)} = ct^{\nu};$$

$$b_1^{(2)} = \frac{\nu(8ct^{\nu+2} + \nu)}{t(16ct^{\nu+2} - \nu^2)}; \quad b_2^{(2)} = ct^{\nu} + \frac{2c\nu t^{\nu}(2 + \nu)}{16ct^{\nu+2} - \nu^2}.$$

Consequently, at k = 2 condition (6.39) is executed. We will check fulfillment of conditions (6.32), considering $t_0 > 0$. The first condition in this case takes the form

$$\int_{t_0}^{\infty} \sqrt{\left| \frac{\nu^2(\nu+2)^2}{4t^2(\nu^2-16ct^{\nu+2})} \right|}\, dt < \infty.$$

It is executed during $\nu > -2$.

The second condition takes the form

$$\int_{t_0}^{\infty} \left| \frac{\sqrt{c}\,\nu(\nu+2)t^{\nu/2}}{\nu^2-16ct^{\nu+2}} \right| dt < \infty.$$

It also is executed during $\nu > -2$.

Consequently, the matrix of coefficients L, interesting us, is diagonal.

Since roots of equations determining the second canonical expansion

$$\zeta_{1,2}^{(1)} = -\frac{\nu}{4t} \mp \frac{1}{2}\sqrt{\frac{\nu^2}{4t^2} - 4ct^{\nu}}$$

satisfy condition (b) of Rapoport's theorem, then, applying it, we will find asymptotic presentation of two linearly independent solutions of equation (4.41),

a) for $c > 0$

$$X_{1,2}(t) = \exp\int\left(-\frac{\nu}{4t} \pm i\sqrt{c}\,t^{\nu/2}\right)dt = t^{-\nu/4}\exp\left(\pm\frac{2i\sqrt{c}}{\nu+2}t^{\nu/2+1}\right);$$

b) for $c < 0$

$$X_{1,2}(t) = \exp\left(\mp\frac{2\sqrt{-c}}{\nu+2}t^{\nu/2+1}\right).$$

§ 3. Asymptotic Presentation of General Solution of an Equation of Oscillations

On the basis of the idea of asymptotic presentation of a particular solution of an equation of free oscillations it is easy to construct the idea of asymptotic presentation of its general solution.

Definition 1. Asymptotic presentation of general solution $x(t)$ of an equation of free oscillations we will call function

$$X(t) = C_1X_1(t) + \ldots + C_nX_n(t), \tag{6.40}$$

where $C_1, \ldots, C_n$ are arbitrary constants; $X_1(t), \ldots, X_n(t)$ are asymptotic presentations of linearly independent particular solutions of equation (0.1).

Let us note that any two asymptotic presentations $X_j(t)$ and $X_j'(t)$ of the same particular solution $x_j(t)$ are connected by dependence

$$X_j(t) = X_j'(t)v_j(t),$$

where $v_j(t)$ is a function satisfying conditions (6.4) during substitution $X = X_j$, $x = X_j'$, $R = v_j$.

Consequently, if there are given certain asymptotic presentations $X_1(t)$, ..., $X_n(t)$ of any particular solutions $x_1(t)$, ..., $x_n(t)$ and during any set of functions $v_j(t)$ ($j = 1, ..., n$), satisfying conditions (6.4) (during above-indicated substitution), it is impossible to obtain linearly dependent set of functions $X_1'(t)$, ..., $X_n'(t)$, then formula (6.40) will determine asymptotic presentation of general soltuion.

If we find asymptotic presentations of particular solutions in the form

$$X_1(t) = \exp\int_{t_0}^{t} z_1^{(k-1)}\,dt, \ldots, X_n(t) = \exp\int_{t_0}^{t} z_n^{(k-1)}\,dt$$

and set of functions $X_1(t)$, ..., $X_n(t)$ cannot be converted by above-indicated method into a linearly dependent system of functions $X_1'(t)$, ..., $X_n'(t)$ then the general solution may be expressed by formula

$$X(t) = C_1\exp\int_{t_0}^{t}(z_1^{(l-1)}+\gamma_1)\,dt+\ldots+C_n\exp\int_{t_0}^{t}(z_n^{(l-1)}+\gamma_n)\,dt, \qquad (6.41)$$

where γ_1, ..., γ_n are arbitrary functions satisfying conditions (6.17).

During $\gamma_1 = ... = \gamma_n = 0$ this formula is turned into formula (4.20) of approximate presentation of general solution. Thus, if there are executed the defined, above-indicated conditions, then one of the asymptotic presentation of the general solution coincides with its approximate presentation of the form (4.20).

Determining asymptotic presentation of general solution of an equation of free oscillations by equality (6.40), we did not limit region of its possible values. Due to this, function $X(t)$ may be complex-valued even in the case when all solutions $x_1(t)$, ..., $x_n(t)$ and all constants C_1, ..., C_n are real. Since we are interested, basically, in real solutions of an equation of oscillations, it is expedient, while considering the general solution of an equation of oscillations in the real region, to determine asymptotic presentation of general solution also in the real region. In connection with this, we will introduce the following definition.

Definition 2. Asymptotic presentation of the general solution of an equation of free oscillations, considered in the real region, we will call real function $X(t)$, connected with asymptotic presentations of linearly independent particular solutions of an equation of oscillations by equality (6.40).

The condition of realness of the general solution and its asymptotic presentation puts a limitation on arbitrariness in the selection of particular solutions x_1, ..., x_n, their asymptotic presentations and constants C_1, ..., C_n. Namely, the systems of the shown magnitudes must consist only of real elements and complex conjugate pairs.

Example: In the case of equation (see example in preceding paragraph)

$$\ddot{x} + ct^{\nu}x = 0,\quad c \neq 0,\quad \nu > -2 \tag{4.41}$$

asymptotic presentations of two linearly, independent particular solutions were found in the form

a) for $c > 0$

$$X_{1,2}(t) = t^{-\nu/4}\exp\left(\mp \frac{2i\sqrt{c}}{\nu+2}\, t^{\nu/2+1}\right),$$

b) for $c < 0$

$$X_{1,2}(t) = \exp\left(\mp \frac{2\sqrt{-c}}{\nu+2}\, t^{\nu/2+1}\right).$$

Using these formulas, one of the asymptotic presentations of general solution we will obtain in the form

a) for $c > 0$

$$X(t) = t^{-\nu/4}\left(C_1 \cos \frac{2\sqrt{c}}{\nu+2}\, t^{\nu/2+1} + C_2 \sin \frac{2\sqrt{c}}{\nu+2}\, t^{\nu/2+1}\right),$$

b) for $c < 0$

$$X(t) = C_1 \exp\left(-\frac{2\sqrt{-c}}{\nu+2}\, t^{\nu/2+1}\right) + C_2 \exp \frac{2\sqrt{-c}}{\nu+2}\, t^{\nu/2+1}.$$

§ 4. Connection of Properties of General Solution of an Equation of Oscillations with Properties of Its Asymptotic Presentation

Let us assume that the asymptotic presentation of the general solution of an equation of oscillations is known. Then each of functions $X_j(t)$ ($j = 1, \ldots, n$), by which it is formed, possesses one of three properties: the function is a) unlimited, b) limited, but nonvanishing, c) vanishing. To establish these properties is very simple:

a) function $X_j(t)$ is unlimited if

$$\overline{\lim_{t\to\infty}} \ln |X_j(t)| = \infty;$$

b) function $X_j(t)$ is limited, but nonvanishing, if

$$\overline{\lim_{t\to\infty}} \ln |X_j(t)| = C.$$

where C is constant;

c) function $X_j(t)$ is vanishing if

$$\overline{\lim_{t\to\infty}} \ln |X_j(t)| = -\infty.$$

We will show that if function $X_j(t)$ possesses one of the shown properties, then particular solution $x_j(t)$, which it presents, possesses the same property.

Indeed, because of conditions (6.5) and (6.7) during $t \to \infty$ between functions $x_j(t)$ and $X_j(t)$ there exists relationship

$$x_j(t) \sim |X_j(t)|^{1+\mathrm{Re}\,o(1)}. \tag{6.42}$$

This relationship leads to equality

$$\mathrm{Re}\ln x_j(t) = [1+\mathrm{Re}\,o(1)]\,\mathrm{Re}\ln X_j(t) + \mathrm{Re}\,o(1). \tag{6.43}$$

Hence, because of identity ln |W| = Re ln W there immediately follows the above-indicated conformity.

The fixed conformity allows, in particular, to determine, with respect to given asymptotic presentation of general resolution, whether oscillations are stable, asymptotically stable, or unstable.

Oscillations are stable if asymptotic presentation of general solution (6.40) is bounded by functions during any finite constants $C_1, \ldots, C_n$. Oscillations are stable asymptotically if in the same conditions asymptotic presentation of the general solution is a vanishing function. Oscillations are unstable if it is possible to indicate such finite values of constants $C_1, \ldots, C_n$, at which asymptotic presentation of the general solution is an unbounded function.

Formulated positions preserve force if one considers general solution of an equation of oscillations in a real region, and the idea of its asymptotic presentation is definitized in accordance with definition 2 of the preceding paragraph.

Now we will assume that asymptotic presentation of the general solution is obtained in the form (6.41). Then functions

$$X_j(t) = \exp\int_{t_0}^{t} (\zeta_j^{(k-1)} + \gamma_j)\,dt \qquad (j = 1, \ldots, n).$$

with help of which it is formed, can be united into two groups, attributing to first group functions for which there occurs

$$\text{a)}\quad \lim_{t\to\infty} \mathrm{Im}\,\mathrm{Ln}\,X_j(t) = \lim_{t\to\infty}\int_{t_0}^{t} \mathrm{Im}\,(\zeta_j^{(k-1)} + \gamma_j)\,dt + 2k\pi = \pm\infty,$$

and to the second group functions for which there occurs

$$\text{b)}\quad \left|\lim_{t\to\infty} \mathrm{Im}\,\mathrm{Ln}\,X_j(t)\right| = \left|\lim_{t\to\infty}\int_{t_0}^{t} \mathrm{Im}\,(\zeta_j^{(k-1)} + \gamma_j)\,dt + 2k\pi\right| < \infty.$$

Other possibilities are excluded since, with respect to construction of canonical expansion, functions Im $\zeta_j^{(k-1)}(t)$ ($j = 1, \ldots, n$) do not change sign, due to which functions Im Ln $X_j(t)$ during $t \to \infty$ can be only nondecreasing, nonincreasing, or approaching final limit.

If function $X_j(t)$ satisfies condition (a) and $\zeta_j = \overline{\zeta}_{j+m}$, then function $X_{j+m}(t)$ also satisfies this condition. With this, functions Im Ln $X_j(t)$ and Im Ln $X_{j+m}(t)$

in any interval (T, ∞) an infinite number of times take values multiple to π.

If function $X_j(t)$ satisfies condition (b), and limit of magnitude Im Ln $X_j(t)$ is different than $k\pi$ (where k is any integer), then there always can be found such value T, at which in interval (T, ∞) this magnitude differs as little as desired from its own limit and, consequently, does not take any zero value or a value multiple to π.

If during fulfillment of condition (b)

$$\lim_{t\to\infty} \operatorname{Im} \operatorname{Ln} X_j(t) = k\pi, \tag{6.44}$$

then, depending upon function $\gamma_j(t)$, approach to limit may be unilateral or bilateral. In the first case there can be found such a value of T at which function $X_j(t)$ in interval (T, ∞) does not take zero values or values multiple of π, in the second case it is impossible.

Let us note that condition (b) can be executed only if there occurs equality

$$\lim_{t\to\infty} \operatorname{Im} \zeta_j = 0. \tag{6.45}$$

Let us consider the general solution of an equation of oscillations in a real region and investigate properties connected with questions of the existence of fluctuating solutions of this equation.

Let us remember that fluctuating real functions or solutions are such real functions or solutions determined in certain interval (t_0, ∞), which in any interval (T, ∞), where $T \geq t_0$, take an infinite set of zero values. Real functions or solutions, not possessing this property, are called <u>nonfluctuating</u>.

Applying definition 2 of an asymptotic presentation of a general solution and considering the properties following from this definition of function $X_j(t)$, we will divide functions $X_1(t)$, ..., $X_n(t)$ into real and complex ones.

From limitedness of function $\zeta_n^{(k-1)}(t)$ in any finite interval, it follows that all real functions $X_j(t)$ do not turn into zero during any sufficiently large value of t, i.e., are nonfluctuating.

If function $X_j(t)$ is complex and $\zeta_j^{(k-1)} = \overline{\zeta}_{j+m}^{(k-1)}$, then function $X_{j+m}(t)$ also is complex, where $X_j = \overline{X}_{j+m}$.

We will consider the sum of $C_jX_j + C_{j+m}$ $(C_j = \overline{C}_{j+m})$.

It, obviously, is real and can be presented in the form

$$C_jX_j + C_{j+m}X_{j+m} = 2|C_j| \exp \operatorname{Re} \ln X_j \cos(\operatorname{Im} \operatorname{Ln} X_j + \varphi_j).$$

where

$$\varphi_j = \arccos\frac{\operatorname{Re} C_j}{|C_j|} = \arcsin\frac{\operatorname{Im} C_j}{|C_j|}.$$

It follows from this that if function $X_j(t)$ satisfies condition (a), then sum $C_jX_j(t) + C_{j+m}X_{j+m}(t)$ is a fluctuating function.

If function $X_j(t)$ satisfies condition (b), then limit of magnitude $\operatorname{Im}\operatorname{Ln} X_j(t) + \varphi_j$ may be unequal or equal to magnitude $k\pi + \pi/2$ (this depends on C_j). In the first case sum $C_jX_j + C_{j+m}X_{j+m}$ is a nonfluctuating function. In the second case it may be both fluctuating and nonfluctuating, depending upon whether there occurs a unilateral or bilateral approach to limit.

Let us assume that the question about which of the real functions $X_j(t)$ or sums of complex conjugate functions $X_j(t)$ and $X_{j+m}(t)$ are fluctuating and which are nonfluctuating is clarified. Then, for clarification of the considered properties of particular solutions we will use equalities (6.43) and equality

$$\operatorname{Im}\operatorname{Ln} x_j(t) = [1 + \operatorname{Re} o(1)]\operatorname{Im}\operatorname{Ln} X_j(t) + 2k\pi + \operatorname{Re} o(1)], \tag{6.46}$$

which, just as the preceding, follows from equality (6.42).

Because of equality (6.43) real asymptotic presentations $X_j(t)$ correspond to nonfluctuating solutions $x_j(t)$.

From equality (6.46) it follows that if complex asymptotic presentation $X(t)$ satisfies condition (a), then analogous condition satisfies solution $x_j(t)$, i.e., $\operatorname{Im}\operatorname{Ln} x_j(t) = \pm\infty$ and, consequently, real solutions $C_jx_j(t) + C_{j+m}x_{j+m}(t)$ are fluctuating. In case (b) particular solutions $C_jx_j(t) + C_{j+m}x_{j+m}(t)$ can be both fluctuating and nonfluctuating independently of what properties sums $C_jX_j(t) + C_{j+m}X_{j+m}(t)$ possess.

In many applications of the theory of free oscillations of linear systems with variable parameters, we are interested in the question of the existence of fluctuating particular solutions of an equation of oscillations. The fixed conformity between properties of particular solutions and their asymptotic presentations allows us to connect this question with the properties of asymptotic presentation of a general solution of an equation of oscillations. On the basis of the above-stated the following conclusions can be made.

If functions $X_1(t)$, ..., $X_n(t)$ are such that at least for one of them is executed condition (a), then equation of oscillations has fluctuating solutions.

If all functions $X_1(t)$, ..., $X_n(t)$ satisfy condition (b), then here are

different possibilities. Equation of oscillations does not have fluctuating solutions if all functions $X_1(t)$, ..., $X_n(t)$ are real, where for all i and j during $i \neq j$.

$$\lim_{t\to\infty} (\operatorname{Re} \ln C_i X_i - \operatorname{Re} \ln C_j X_j) =$$
$$= \int_{t_0}^{t} (\xi_i^{(n-1)} - \xi_j^{(n-1)})\, dt + \ln|C_i| - \ln|C_j| = \pm\infty.$$

If during fulfillment of condition (b) and realness of all functions $X_j(t)$ the mentioned additional condition is not executed, then fluctuating solutions exist in the case when for any i and j, magnitudes

$$\operatorname{Re} \ln x_i - \operatorname{Re} \ln x_j + C$$

(C is any real constant) during $t \geq T$, where T is sufficiently great, are sign-alternating, and do not exist otherwise. To establish which of these possibilities in reality takes place by the form of function $X_i(t)$ and $X_j(t)$ is impossible.

If during fulfillment of condition (b) among functions $X_1(t)$, ..., $X_n(t)$ there are complex ones and for all i and j ($i \neq j$) there is executed condition

$$\lim_{t\to\infty} (\operatorname{Re} \ln X_i - \operatorname{Re} \ln X_j) = \pm\infty,$$

if $X_i \neq \overline{X}_j$, then fluctuating solutions exist when for certain i and j, at which $X_i = \overline{X}_j$, during certain C, magnitude $\operatorname{Im} \operatorname{Ln} x_i + C$ during $t \geq T$ is sign-alternating; and they do not exist if such values of i, j, and C are lacking. To establish which of these possibilities in reality takes place, by form of function $X_i(t)$ and $X_j(t)$ it is impossible.

Example: In the example given in the preceding paragraph are found asymptotic presentations of the general solution of equation

$$\ddot{x} + ct^{\nu}x = 0, \quad c \neq 0, \quad \nu > -2. \tag{4.41}$$

Functions $X_1(t)$ and $X_2(t)$, by which it is formulated, have the form

a) for $c > 0$

$$X_{1,2}(t) = t^{-\nu/4} \exp\left(\pm \frac{2i\sqrt{c}}{\nu+2} t^{\nu/2+1}\right).$$

b) for $c < 0$

$$X_{1,2}(t) = t^{-\nu/4} \exp\left(\pm \frac{2\sqrt{-c}}{\nu+2} t^{\nu/2+1}\right).$$

In case (a)

$$\lim_{t\to\infty} \operatorname{Re} \ln X_{1,2} = \lim_{t\to\infty} \left(-\frac{\nu \ln t}{4}\right) = \begin{cases} = \infty & \text{for } \nu < 0, \\ = 0 & \text{for } \nu = 0, \\ = -\infty & \text{for } \nu > 0. \end{cases}$$

Consequently, during c > 0 oscillations are unstable if $\nu < 0$; stable if $\nu = 0$ and asymptotically stable if $\nu > 0$.

Since in the considered case

$$\lim_{t\to\infty} \operatorname{Im} \operatorname{Ln} X_{1,2}(t) = \pm ,$$

all particular solutions are fluctuating.

In case (b) functions $X_1(t)$ and $X_2(t)$ are real where there is executed the above-indicated additional condition. Consequently, all particular solutions are nonfluctuating.

Further, since

$$\lim_{t\to\infty} \ln X_1(t) = -\lim_{t\to\infty} \ln X_2(t) = -\infty,$$

there exist two linearly independent solutions, one of which is a vanishing function and the other unlimited. Consequently, during c < 0 oscillations are unstable, independently of values ν.

CHAPTER VII

FREE OSCILLATIONS OF LINEAR SYSTEMS WITH PERIODICALLY VARIABLE PARAMETERS

Linear systems with periodically variable parameters are linear systems whose equations of free oscillations are equations with periodic coefficients, i.e., have as coefficients $b_1(t)$, ..., $b_n(t)$ periodic functions of one and the same period.

Methods of analysis of an equation of free oscillations, well-developed in Chapters IV-VI, during application to equations of different classes, give in some cases more effective results, in other cases less effective results. The forms of equations for which the presented methods are in general comparatively little effective include equations with periodic coefficients.

Solutions of equations with periodic coefficients, however, possess specific properties, using which, it is possible to strengthen the above-stated methods.

§ 1. Certain Remarks About Equations of Free Oscillations With Periodic Coefficients and, Corresponding to Them, Systems of Equations Relative to Canonical Components

Let us assume that equation (0.1) has as coefficients b_1, ..., b_n periodic functions of t with the same period Ω, i.e., functions $b_i(t)$ ($i = 1, ..., n$) satisfying condition

$$b_i(t+\Omega)=b_i(t) \qquad (i=1, \ldots, n). \tag{7.1}$$

The form of this equation will not be changed if variable t is replaced by magnitude $t+\Omega$. Consequently, if general solution of equation (0.1) in interval[1] (t_0, $t_0 + \Omega$)

[1]Here and subsequently the positivity of magnitude Ω is assumed.

is known, then also general solution of this equation in any interval ($t_0 + k\Omega$, $t_0 + k\Omega + \Omega$) is known (where k is a natural number), and any particular solution determined in interval (t_0, $t_0 + \Omega$) may be continued in any interval as large as desired, adjoint on the right to interval (t_0, $t_0 + \Omega$).

Everything said above it is possible to repeat for systems of linear equations with periodic coefficients, real or complex, in particular for systems of equations relative to canonical components, obtained as a result of canonical expansions of solution of equation (0.1) with coefficients satisfying condition (7.1). The presence of analogous properties of solutions for the latter systems is connected with the fact that their coefficients also are periodic functions t of period Ω. This property is possessed by all systems of equations relative to canonical components independently of the structure of canonical expansion.

§ 2. Structure of Solutions of a System of Equations Relative to Canonical Components

In this paragraph we will consider properties of solutions of a system of equations (2.34) connecting canonical components $y_1, \ldots, y_n$ of solution of equation (0.1) during unmodulated structure of expansion, considering that the whole course of reasonings and conclusions completing it can, without reservations, refer to a system of equations obtained during modified structure of canonical expansion and also to systems of equations relative to canonical components $z_1, \ldots, z_n$.

Let us assume that

$$y_{1i}(t), \ldots, y_{ni}(t) \quad (i=1, \ldots, n) - \qquad (7.2)$$

of n linearly independent solutions of system of equations relative to canonical components. Then set of functions

$$y_{1i}(t+\Omega), \ldots, y_{ni}(t+\Omega) \quad (i=1, \ldots, n) \qquad (7.3)$$

also are solutions of shown systems and, consequently, each of functions $y_{ji}(t+\Omega)$ ($j = 1, \ldots, n$) may be linearly expressed through functions $y_{ji}(t)$ ($i = 1, \ldots, n$).

We will write

$$y_{ji}(t+\Omega) = a_{1i} y_{j1}(t) + \ldots + a_{ni} y_{jn}(t) \quad (j=1, \ldots, n), \qquad (7.4)$$

where a_{ji} ($j, i = 1, \ldots, n$) are certain constants.

Let us assume that there exists nontrivial solution of system (2.34)

$$y_{1s}(t), \ldots, y_{ns}(t),$$

in general, differing from solutions (7.2) and possessing property

$$y_{jn}(t+\Omega)=\varkappa y_{jn}(t) \quad (j=1,\ldots,n). \tag{7.5}$$

We will express this solution through solutions (7.2)

$$\begin{gathered} y_{jn}(t)=\beta_1 y_{j1}(t)+\ldots+\beta_n y_{jn}(t), \\ y_{jn}(t+\Omega)=\beta_1 y_{j1}(t+\Omega)+\ldots+\beta_n y_{jn}(t+\Omega) \quad (j=1,\ldots,n). \end{gathered} \tag{7.6}$$

Here $\beta_1, \ldots, \beta_n$ are certain constants.

On the basis of equations (7.4)-(7.6), we will obtain

$$\begin{gathered} \beta_1[a_{11}y_{j1}(t)+\ldots+a_{n1}y_{jn}(t)]+\ldots+\beta_n[a_{1n}y_{j1}(t)+\ldots+a_{nn}y_{jn}(t)]= \\ =\varkappa[\beta_1 y_{j1}(t)+\ldots+\beta_n y_{jn}(t)] \quad (j=1,\ldots,n). \end{gathered} \tag{7.7}$$

Since functions $y_{j1}(t), \ldots, y_{jn}(t)$ are linearly independent, then equations (7.7) are satisfied independently of values t only in the case when sums of coefficients during each function $y_{ji}(t)$ are equal to zero. If follows from this that

$$\left.\begin{array}{l} (a_{11}-\varkappa)\beta_1+\ldots+a_{1n}\beta_n=0, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\beta_1+\ldots+(a_{nn}-\varkappa)\beta_n=0. \end{array}\right| \tag{7.8}$$

System (7.8) it is possible to consider as a system of linear uniform algebraic equations relative to unknowns $\beta_1, \ldots, \beta_n$. It allows nontrivial solution if its determinant is equal to zero. Equating the latter to zero, we will obtain equation

$$\|a_{ij}-\delta_{ij}\varkappa\|=\varkappa^n+A_1\varkappa^{n-1}+\ldots+A_n=0, \tag{7.9}$$

which is called <u>characteristic</u>. Coefficients of this equation are <u>invariant relative to selected system of linearly independent solutions (7.2) and selected solution possessing property (7.5)</u> [6], i.e., they are simply determined by coefficients of system of equations (2.34).

Analytic dependences between coefficients of characteristic equation $A_1, \ldots, A_{n-1}$ and coefficients of its corresponding system of equations up to now are not clarified. The very fact of their existence is not clear. Regarding, however, absolute term of characteristic equation A_n, it is connected with coefficients of the system, placed on main diagonal of matrix $\lambda_1 + g_{11}, \ldots, \lambda_n + g_{nn}$, corresponding to it, by simple dependence [6].

In order to obtain this dependence, we will constitute determinant from system of fundamental solutions (7.2)

$$\Delta=\det\|y_{ji}\|_1^n$$

and, according to the formula of Liouville, we will write

$$\Delta(t+\Omega)=\Delta(t)\exp\int_0^\Omega\sum_{i=1}^n(\lambda_i+g_{ii})\,dt.$$

Since, because of relationship (7.4), determinant $\Delta(t+\Omega)$ is equal to the product of determinant $\Delta(t)$, by determinant det $\|a_{ji}\|_1^n$, then the written equality takes the form

$$\det\|a_{ji}\|_1^n=\exp\int_0^\Omega\sum_{i=1}^n(\lambda_i+g_{ii})\,dt.$$

Because of evident relationship

$$A_n=(-1)^n\det\|a_{ji}\|_1^n$$

we will obtain

$$A_n=(-1)^n\exp\int_0^\Omega\sum_{i=1}^n(\lambda_i+g_{ii})\,dt.$$

In accordance with formulas for coefficients g_{ii} [see explanation to equation (2.34)] and equality

$$\sum\lambda_i=-b_1$$

this dependence takes the form

$$A_n=(-1)^n\exp\left[-\int_0^\Omega\left(b_1+\frac{\dot{W}}{W}\right)dt\right].$$

Since

$$\int_0^\Omega\frac{\dot{W}}{W}\,dt=W(\Omega)-W(0)=0,$$

the last formula is simplified to the form

$$A_n=(-1)^n\exp\left(-\int_0^\Omega b_1\,dt\right). \qquad (7.9a)$$

It is possible to show that formula (7.9a) also is valid for the absolute term of the characteristic equation corresponding to any system of equations relative to canonical components $z_1, \ldots, z_n$.

Let us assume that $\varkappa_1, \ldots, \varkappa_n$ are roots of equation (7.9). Then these magnitudes are those values of $\varkappa$ with which correspond certain particular solutions of system (2.34), possessing property (7.5). It is obvious that there cannot exist other particular solutions possessing this property during other values of $\varkappa$, since system (7.8) will satisfy only roots of equation (7.9).

Among roots $\varkappa_1, \ldots, \varkappa_n$ there cannot be zero. This could take place only in the case when determinant det $\|a_{ij}\|$ was equal to zero. But the latter is excluded since in this case, because of equations (7.4), solutions (7.3), and consequently also (7.2), cease to be linearly independent.

From the general theory of linear, uniform, differential equations with periodic coefficients [6.24], it follows that every root of characteristic equation (7.9) corresponds to solution of system (2.34) form

$$y_{1i} = \varkappa_i^{t/\Omega} \varphi_{1i}(t), \ldots, y_{ni} = \varkappa_i^{t/\Omega} \varphi_{ni}(t), \tag{7.10}$$

where $\varphi_{ji}(t)$ $(j = 1, \ldots, n)$ are continuous, limited, periodic functions t with period Ω, among which at least one is not equal to zero identically. Therefore, if the characteristic equation does not have multiple roots, then, considering all roots, we will obtain n solutions of such form, and these solutions will be linearly independent.

From the above-mentioned theory, it also follows that if characteristic equation has multiple roots, then number of linearly independent solutions of form (7.10) is equal to number of different roots, where each of different roots enters into formulas only for one solution. Every μ-multiple root $\varkappa_1$, besides solution of form (7.10), corresponds to $\mu - 1$ linearly independent solution of another form, where these solutions can be selected in such a way that solution of form (7.10) is impossible to obtain as a result of their combining. Selected, as is shown, it is possible to record solution in the form (7.10), if symbols $\varphi_{ji}(t)$ $(j = 1, \ldots, n)$ are designated not periodic functions but functions of the form

$$\varphi_{ji}(t) = \psi_{ji}^{(0)}(t) + t\psi_{ji}^{(1)}(t) + \ldots + t^m \psi_{ji}^{(m)}(t), \tag{7.11}$$

where all $\psi_{ji}^{(k)}(t)$ $(k = 0, 1, \ldots, m)$ are continuous, limited periodic functions of t with period Ω, where $\psi_{ji}^{(m)}(t) \not\equiv 0$, and $m = 1, 2, \ldots, \mu - 1$.

We will define magnitudes $\alpha_1, \ldots, \alpha_n$, corresponding to roots $\varkappa_1, \ldots, \varkappa_n$ by formulas

$$\alpha_i = \frac{1}{\Omega} \ln \varkappa_i \qquad (i = 1, \ldots, n). \tag{7.12}$$

These magnitudes are called <u>characteristic indices</u> of system of equations corresponding to given characteristic equation.

Using formulas (7.12) formulas (7.10) we will present in the form

$$y_{1i} = \varphi_{1i}(t) \exp \alpha_i t, \ldots, y_{ni} = \varphi_{ni}(t) \exp \alpha_i t. \tag{7.13}$$

This is valid also for equations relative to canonical components $z_1, \ldots, z_n$.

§ 3. Periodic Properties of Phase Coefficients, Logarithmic Derivatives of Canonical Components and Logarithmic Derivative of Norm of Solution of a System of Equations Relative to Canonical Components

Let us assume that nontrivial particular solution of a system of equations relative to canonical components

$$y_{1i}(t), \ldots, y_{ni}(t)$$

or

$$z_{1i}(t), \ldots, z_{ni}(t)$$

has the form (7.10), where all functions $\varphi_{ji}(t)$ are periodic. We will establish form of its corresponding norm of solution and phase coefficients.

From determination of norm of solution [equation (3.6)] it follows that in the considered case it may be presented in the form

$$r_i(t) = \sqrt{\varphi_{1i}\bar{\varphi}_{1i} + \cdots + \varphi_{ni}\bar{\varphi}_{ni}} \exp \operatorname{Re} \alpha_i t = \varphi_i(t) \exp \operatorname{Re} \alpha_i t, \tag{7.14}$$

where $\varphi_i(t)$ is continuous, differentiable, limited periodic function t with period Ω. Because of properties of norm of solution and condition $\varkappa_1 \neq 0$ (see § 2), function $\varphi_i(t)$ does not turn into zero during any value of t from interval $(0, \omega)$.

Differentiating equality (7.14), we will find

$$\dot{r}_i = (\varphi_i \operatorname{Re} \alpha_i + \dot{\varphi}_i) \exp \operatorname{Re} \alpha_i t.$$

Comparing the two last equalities, we will obtain

$$\frac{\dot{r}_i}{r_i} = \frac{\varphi_i \operatorname{Re} \alpha_i + \dot{\varphi}_i}{\varphi_i}. \tag{7.14a}$$

From equality (7.14a), it follows that <u>during periodic (with period Ω) functions $\varphi_{ji}(t)$ the logarithmic derivative of the norm of considered i-th solution of a system of equations relative to canonical components is a continuous, limited, periodic function with the same period.</u>

On the basis of formulas (7.13) and (7.14) for phase coefficients, we will obtain

$$e_{ji}(t) = \frac{\varphi_{ji}(t)}{\varphi_i(t)} \exp(\sqrt{-1} \operatorname{Im} \alpha_i t). \tag{7.15}$$

Consequently, during $\operatorname{Im} \alpha_i \neq 0$, phase coefficients $e_{ji}(t)$ $(j = 1, \ldots, n)$ can be presented in the form of products of pairs of continuous, bounded functions with periods Ω and $2\pi/\operatorname{Im} \alpha_1$. During $\operatorname{Im} \alpha_1 = 0$, i.e., in the case when characteristic index is real, $\exp(\sqrt{-1} \operatorname{Im} \alpha_1 t) = 1$ and phase coefficients are continuous, limited, periodic functions with period Ω.

Using formula (7.15), we will constitute expression for squares of moduli of phase coefficients

$$|e_{ji}(t)|^2 = e_{ji}(t)\,\bar{e}_{ji}(t) = \frac{\varphi_{ji}(t)\bar{\varphi}_{ji}(t)}{\varphi_i(t)\bar{\varphi}_i(t)} = \frac{|\varphi_{ji}(t)|^2}{|\varphi_i(t)|^2} \quad (j=1,\ldots,n). \tag{7.16}$$

From formula (7.16) it follows: if functions $\varphi_{ji}(t)$ are periodic with period Ω, then squares of moduli of phase coefficients are continuous, limited, periodic functions of the same period.

We will establish now periodic properties of logarithmic derivatives of canonical components.

Obviously, if $y_{ji}(t) \not\equiv 0$ or $z_{ji}(t) \not\equiv 0$, then during all values of t, for which $y_{ji}(t) \neq 0$ or $z_{ji}(t) \neq 0$,

$$\left.\begin{aligned}\frac{d\ln y_{ji}}{dt} = \frac{\dot{y}_{ji}}{y_{ji}} = \\ \frac{d\ln z_{ji}}{dt} = \frac{\dot{z}_{ji}}{z_{ji}} = \end{aligned}\right\} = \frac{\dot{r}_i e_{ji} + r_i \dot{e}_{ji}}{r_i e_{ji}} = \frac{\dot{r}_i}{r_i} + \frac{\dot{e}_{ji}}{e_{ji}}.$$

According to formula (7.14a)

$$\frac{\dot{r}_i}{r_i} = \frac{\varphi_i \operatorname{Re} a_i + \dot{\varphi}_i}{\varphi_i}.$$

Because of formulas (7.15)

$$\frac{\dot{e}_{ji}}{e_{ji}} = \frac{\frac{\varphi_{ji}}{\varphi_i}\sqrt{-1}\,\operatorname{Im} a_i + \frac{\dot{\varphi}_{ji}\varphi_i - \varphi_{ji}\dot{\varphi}_i}{\varphi_i^2}}{\varphi_{ji}/\varphi_i} =$$

$$= \frac{\varphi_{ji}\sqrt{-1}\,\operatorname{Im} a_i + \dot{\varphi}_{ji} - \frac{\varphi_{ji}\dot{\varphi}_i}{\varphi_i}}{\varphi_{ji}} \quad (j=1,\ldots,n). \tag{7.18}$$

Formulas (7.18) have meaning only for those values of t at which $\varphi_{ji} \neq 0$.

From formulas (7.14a), (7.17), and (7.18), it follows that logarithmic derivatives of canonical components not equal identically to zero are periodic functions t of period Ω, determined everywhere besides those points of the investigated interval in which $\varphi_{ji}(t) = 0$.

Found properties of phase coefficients and logarithmic derivatives of canonical components present essential interest later. The basic result, which we obtained above and use subsequently, can be formulated as follows.

A system of equations relative to canonical components of solution of equation (7.1) with coefficients which are periodic (with general period Ω) functions of t, has from one to n linearly independent solutions whose y are squares of moduli of

phase coefficients; logarithmic derivatives of canonical components and the logarithmic derivative of the norm during any (allowed by considered equation) canonical expansion of its solution are periodic functions of t with the same period.

Let us consider now the particular solution of a system of equations relative to canonical components

$$y_{1i}, \ldots, y_{ni} \text{ or } z_{1i}, \ldots, z_{ni},$$

having the form (7.10) under the condition that at least one of the functions $\varphi_{ji}(t)$ is nonperiodic form (7.11).

Norm of solution $r_i(t)$ in this case may be presented in the form (7.14), but function $\varphi_i(t)$ here is no longer periodic; φ_i^2 has the form of functions (7.11). Just as in the preceding case, function $\varphi_i(t)$ does not turn into zero during any value of t from interval (0, ∞).

We will multiply the numerator and denominator of fraction (7.14a) by φ_i, presenting them in the form of polynomials of type (7.11). Obviously, degree of polynomial $\dot{\varphi}_i\varphi_i$ cannot be higher than degree of polynomial φ_i^2. Consequently, degree of polynomial representing numerator of the right side of equality (7.14a) is not higher than degree of polynomial representing its denominator. And hence it follows that logarithmic derivative of norm of solution during $t \to \infty$ either approaches a periodic function equal to the ratio of coefficient during highest power of t of the polynomial representing the numerator to an analogous coefficient of the polynomial representing the denominator (case of equal powers of polynomials), or approaches zero (case of unequal powers of polynomials). In the usual language of analysis it is possible to formulate this position in the following form: for any positive value of ε as small as desired it is always possible to select such $t = T$ that for all $t > T$

$$\left|\frac{\dot{r}_i(t)}{r_i(t)} - R_i(t)\right| < \varepsilon,$$

where $R_i(t)$ is a periodic function equal to the quotient from division of coefficient during power t^{m_i} of polynomial $\varphi_i^2 \operatorname{Re} s_i + \dot{\varphi}_i\varphi_i$ by coefficient with that same power of polynomial φ^2; m_i is power of polynomial φ_i^2.

Phase coefficients in the considered case also can be represented in the form (7.15), but the first factor here is already nonperiodic.

We will investigate properties of functions $\varphi_{ji}(t)/\varphi_i(t)$ during $t \to \infty$.

Let us note that, because of relationships (7.14), highest power t for polynomial

$\varphi_i^2(t)$ is no lower than highest power t for polynomial $[\varphi_{ji}^2(t)]$. Therefore, for any positive value ε as small as desired it is always possible to select such $t = T$ that for all $t \geq T$

$$\left|\frac{\varphi_{ji}(t)}{\varphi_i(t)} - \theta_{ji}(t)\right| < \varepsilon,$$

where $\theta_{ji}(t)$ is a periodic function equal to the quotient from division of coefficient during power t^{m_i} of polynomial $\varphi_{ji}(t)$ by square root of coefficient during power t^{2m_i} of polynomial $\varphi_i^2(t)$, where under $2m_i$ here is understood the biggest exponent of polynomial $\varphi_i^2(t)$. Found dependences between functions $\dot{r}_i(t)/r_i(t)$ and $R_i(t)$, $\varphi_{ji}(t)/\varphi_i(t)$ and $\theta_{ji}(t)$ can be recorded in the form

$$\left.\begin{aligned} &\lim_{T\to\infty} \frac{\dot{r}_i(t)}{r_i(t)} = R_i(t), \\ &\lim_{T\to\infty} \frac{\varphi_{ji}(t)}{\varphi_i(t)} = \theta_{ji}(t) \qquad (T \leqslant t < \infty). \end{aligned}\right| \tag{7.19}$$

Obviously, for squares of moduli of phase coefficients in the considered case, formulas (7.16) also are valid. Because of formula (7.19) from formulas (7.16) there follows

$$\lim_{T\to\infty} |e_{ji}(t)|^2 = |\theta_{ji}(t)|^2 \qquad \left(\begin{matrix} j = 1, \ldots, n; \\ T \leqslant t < \infty \end{matrix}\right). \tag{7.19a}$$

Logarithmic derivatives of canonical component in the considered case are connected with functions $\varphi_i(t)$ and $\varphi_{ji}(t)$ also by formulas (7.14a), (7.17), and (7.18). If one were to present magnitudes (7.14a) and (7.18) in the form of a relation of polynomials of type (7.11), then it is simple to perceive that highest power t in the denominator of each formula is no lower than highest power t in its numerator. Therefore, for any positive values ε_1 and ε_2 as small as desired it is always possible to select such T that logarithmic derivative of j-th canonical component coincides with an accuracy up to ε_1 with a certain periodic function during all $t \geq T$, excluding intervals $(t_k - \varepsilon_2, t_k + \varepsilon_2)$, where t_k are those values of t at which function $\varphi_{ji}(t)$ turns into zero.

On the basis of that presented, it is possible to generalize earlier the obtained result in the following manner.

Theorem. A system of equations relative to canonical components of solution of an equation of free oscillations with coefficients which are differentiable by real

periodic functions of t with general period Ω has n linearly independent solutions, for which squares of moduli of phase coefficients, logarithmic derivatives of canonical components, and logarithmic derivative of norm of solution during any canonical expansion of solution, allowed by the considered equation, are either periodic functions of t of the same period or functions approaching at $t \to \infty$ periodic $|\theta_{ji}(t)|^2 = |\theta_{ji}(t + \Omega)|^2$, $\eta_{ji}(t) = \eta_{ji}(t + \Omega)$, $R_i(t) = R_i(t + \Omega)$, so that

$$\lim_{T\to\infty} \frac{\varphi_{ji}(t)}{\varphi_i(t)} = \theta_{ji}(t),$$

$$\left.\begin{array}{l}\lim\limits_{T\to\infty} \dfrac{\dot{y}_{ji}(t)}{y_{ji}(t)} = \\ \lim\limits_{T\to\infty} \dfrac{\dot{z}_{ji}(t)}{z_{ji}(t)} = \end{array}\right\} = \eta_{ji}(t),$$

$$\lim_{T\to\infty} \frac{\dot{r}_i(t)}{r_i(t)} = R_i(t) \qquad \begin{pmatrix} j=1,\ldots,n; \\ T<t<\infty \end{pmatrix};$$

with this, limiting relationships for logarithmic derivatives of canonical components are determined everywhere except, possibly, those points where canonical components turn into zero.

§ 4. Stability of Oscillations

In § 2 there was determined analytic form of n linearly independent particular solutions of a system of equations relative to canonical components. Each such solution was set in conformity with characteristic index α_i. From form of solutions it is obvious that asymptotic appraisals of the numerical growth of canonical components can be, in the closest form, connected with the real parts of characteristic indices.

Asymptotic properties of canonical components interest us to the degree that they characterize the asymptotic behavior of solution of equation (0.1). The dependence between the asymptotic properties of canonical components and solution of equation (0.1), if asymptotic properties of the first are investigated on the basis of appraisal of characteristic indices, allows us to establish the following lemma.

Lemma. Let us assume that system of equations

$$\left.\begin{array}{l} \dot{x}_1 = x_2, \\ \dot{x}_2 = x_3, \\ \ldots\ldots \\ \dot{x}_{n-1} = x_n, \\ \dot{x}_n = -b_n x_1 - b_{n-1} x_2 - \ldots - b_1 x_{n-1} \end{array}\right\} \tag{7.20}$$

with continuous, limited, periodic real functions t with period Ω will be converted

with the help of linear substitution

$$u_i = q_{i1}x_1 + \ldots + q_{in}x_n \qquad (i = 1, \ldots, n), \tag{7.21}$$

processing the following properties: all coefficients q_{ij} are continuous and limited periodic functions of t with period Ω; their first derivatives are functions of the same character; magnitude reverse to determinant composed of these coefficients is a bounded function of t. Then the system of real parts of n characteristic indices of the converted system will be identical with the system of real parts of n characteristic indices of the initial system.

This lemma is the result of a more general position proven by A. M. Lyapunov [6]. From it, it follows that the system of real parts of n characteristic indices of a system of equations, obtained as a result of any canonical expansion of the solution of equation (0.1) with periodic coefficients, is identical with the system of real parts of n characteristic indices of system (7.20). Since

a) system of equations (7.20) has n linearly independent solutions of form (7.13), where $\varphi_{ji}(t)$ are either periodic functions or functions of form (7.11),

b) these solutions correspond to linearly independent functions $x_{1i}(t), \ldots, x_{ni}(t)$,

c) $x_{1i} \not\equiv 0$ if at least for one j occurs inequality $x_{ji} \not\equiv 0$ [the last two properties follow from relations $x_{i-1} = \dot{x}_i$ $(i = 2, \ldots, n)$], then from the above said it follows that there exists a fundamental system of solutions of equation (0.1), consisting of functions $x^{(i)}(t)$ $(i = 1, \ldots, n)$ of the form

$$x^{(i)}(t) = \chi_i(t) \exp \alpha_i t. \tag{7.22}$$

With this, functions $\chi_i(t)$ are either periodic or of the form (7.11), and α_i are characteristic indices of system (7.20), which, possibly, are different than characteristic indices of a system of equations relative to canonical components but have with them identical real parts.

The fixed form of the fundamental system of solutions of equation (0.1) allows us to make the conclusion which we will formulate in the form of the following theorem.

Theorem. If a system of equations relative to any canonical components of an equation of free oscillations with periodic coefficients has only characteristic indices whose real parts are negative, then free oscillations are stable asymptotically. If among the characteristic indices of the mentioned system there are those whose real parts are positive, then free oscillations are unstable. If one of

the characteristic indices has zero real part but the real parts of all the others are negative, then free oscillations are stable but not asymptotically.

This theorem, which essentially constitutes one of the modifications of the known theorem of A. M. Lyapunov [6], reduces the problem of research of stability of oscillations to the problem of determining properties of characteristic indices of a system of equations relative to canonical components. This problem, which also is very complicated, is not considered in this work.

The theorem embraces all possible cases besides one, and that is the case when two or more characteristic indices of a system of equations relative to canonical components have zero real parts. In this case, in accordance with form of functions $\chi_1(t)$, there can take place both stability (but not asymptotic) and instability.

We will consider methods of research of stability of oscillations of a system with periodically variable parameters, connected with appraisal of asymptotic behavior of norm of solution of a system of equations relative to canonical components. For determination of sufficient conditions of stability with the help of such an appraisal, in the considered case we will apply the general method presented in §§ 2-3 Chapter V; however, specific character of periodically variable coefficients allows us to expand the possibility of research both by means of modification of canonical expansions and by means of calculation of periodic properties of squares of moduli of phase coefficients or logarithmic derivative of norm of solution.

In Chapter II as functions $\lambda_1(t)$, ..., $\lambda_n(t)$ and $\zeta_1^{(k)}(t)$, ..., $\zeta_n^{(k)}$, determining canonical expansions, there were selected roots of certain algebraic equations. There were given general foundations of such selection. If coefficients of equation (0.1) are periodic and (a sufficient number of times) differentiable functions, then it is possible to construct other constructions of canonical expansions and to base the principle of their structure. These constructions coincide with constructions of canonical expansions of the second form with respect to structure of expansion [see system of equations (2.39)], but differ from them by method of determining functions $\zeta_1^{(k)}$, ..., $\zeta_n^{(k)}(t)$.

The idea on which is based new construction is the following. Since logarithmic derivatives of canonical components are periodic or approaching periodic (during $t \to \infty$) functions then with successful selection of functions $\zeta_1(t)$, ..., $\zeta_n(t)$, transforming matrix of coefficients of system of equations relative to canonical components to diagonal, these functions are logarithmic derivatives of corresponding canonical components and, consequently, also have to be periodic or approaching

periodic. Let us assume that these functions or their limiting images (if functions are approaching periodic) satisfy conditions with which they can be expanded into converging Fourier series.[1] We will write their expansion:

$$\left.\begin{array}{r}\zeta_i(t)= \\ \lim\limits_{\substack{t\to\infty \\ T<t<\infty}} \zeta_i(t)=\end{array}\right\} = \sigma_{0i} + \sigma'_{1i}\cos\frac{2\pi t}{\Omega} + \sigma''_{1i}\sin\frac{2\pi t}{\Omega} +$$

$$+ \sigma'_{2i}\cos\frac{4\pi t}{\Omega} + \sigma''_{2i}\sin\frac{4\pi t}{\Omega} + \ldots \qquad (i=1,\ldots,n). \tag{7.23}$$

Now we will turn to formulas for coefficients h_{ij} of the system of equations relative to canonical components [see, for instance, explanation to equations (2.47)] and let us note that they turn into zero if there are executed conditions

$$(\zeta_i + D)^{n-1}\zeta_i + b_1(\zeta_i + D)^{n-2}\zeta_i + \ldots + b_n = 0 \tag{7.24}$$

$$(i=1,\ldots,n).$$

Putting for arbitrary number i, instead of function $\zeta_i(t)$, the right side of i-th equation of system (7.23), we will obtain an equation into which, as unknowns, enter coefficients σ_{0i}, σ'_{1i}, σ''_{1i}, ...

With the help of basic formulas of trigonometry, it may be given the form

$$\tau_{0i} + \tau'_{1i}\cos\frac{2\pi t}{\Omega} + \tau''_{1i}\sin\frac{2\pi t}{\Omega} + \tau'_{2i}\cos\frac{4\pi t}{\Omega} + \tau''_{2i}\sin\frac{4\pi t}{\Omega} + \ldots = 0, \tag{7.25}$$

where τ_{0i}, τ'_{1i}, τ''_{1i}, τ'_{2i}, τ''_{2i} are certain algebraic functions of coefficients σ_{0i}, σ'_{1i}, σ''_{1i} ... and of coefficients of equation (0.1).

Problem of determining asymptotic form of a fundamental system of solutions of equation (0.1) would be completely solved if we found n different systems of values of coefficients σ_{0i}, σ'_{1i}, σ''_{1i}, ... as solutions of a system of infinite number of equations

$$\left.\begin{array}{l}\tau_{0i}=0, \\ \tau'_{ki}=0, \\ \tau''_{ki}=0 \\ (k=1,2,\ldots).\end{array}\right\} \tag{7.26}$$

However, on the path of solution of system (7.26) there appear insuperable difficulties.

---

[1]As is known, periodic, complex-valued function of real variable t with real period Ω may be expanded into a converging Fourier series if in arbitrary interval $(t, t+\Omega)$ its real and imaginary parts are continuous and either do not have extrema or have a finite number of them.

We will be limited now by the realizable. We will replace exact expansion (7.23) by approximate

$$\left.\begin{array}{r}\zeta_i(t)\approx\\ \lim\limits_{\substack{T\to\infty\\ T<t<\infty}}\zeta_i(t)\approx\end{array}\right\}\approx\sigma_{0i}+\sum_{k=1}^{m}\left(\sigma'_{ki}\cos\frac{2k\pi t}{\Omega}+\sigma''_{ki}\sin\frac{2k\pi t}{\Omega}\right) \quad (7.27)$$

and system (7.26) by system

$$\left.\begin{array}{l}\tau_{0i}=0,\\ \tau'_{ki}=0,\\ \tau''_{ki}=0,\\ (k=1,2,\ldots,m).\end{array}\right\} \quad (7.28)$$

If one were to assume that system (7.28) has n different solutions (this possibility is easy to see in the simplest case when it degenerates into one equation: τ_{0i} 0) and to select as functions $\zeta_i(t)$ the right sides of formulas (7.27), then one should expect that matrix of coefficients of system of equations relative to canonical components will be, in a certain meaning, close to diagonal.

Being based on the presented idea, we will select as functions $\zeta_1^{(0)}(t),\ldots,\zeta_n^{(0)}(t)$, determining first canonical expansion, constants σ_{01}, ..., σ_{0n} which are roots of equation $\tau_{0i} = 0$.

Second canonical expansion we will determine, considering in equation (7.27) $m = 1$ and determining coefficients σ_{0i}, σ'_{1i}, σ''_{1i} ($i = 1, \ldots, n$) from system (7.28) where we will also assume $m = 1$.

Analogously, we will determine the next canonical expansions. As in the case of the above-considered canonical expansions, we will allow only such canonical expansions at which determinant of system of equations determining expansion does not turn into zero during any t from a sufficiently remote interval (T, ∞).

These expansions can be applied for research of free oscillations in a finite interval since all the noted-earlier properties of norm of oscillations and phase coefficients (see Chapter II) in these expansions are kept. However, in the presence of multiple characteristic indices for problems of a finite interval they can appear little effective. For analysis of asymptotic properties of free oscillations, the shown constructions can be applied with the same reservation, in spite of the fact that limiting form (at $t \to \infty$) of unknown functions $\zeta_i(t)$ is periodic.

It is natural to expect that effectiveness of the obtained, sufficient conditions of stability increases with increase in number of canonical expansion. However, simultaneously increases also complexity of calculations. If one were to

be limited by a certain definite complexity of calculation, in a number of cases such deviation can be expedient from given methods of determining functions $\zeta_1^{(k)}(t), \ldots, \zeta_n^{(k)}(t)$, at which some coefficients σ_{01}, σ_{11}', σ_{11}'' are selected from some additional conditions obtained by means of simplification of certain equations of system (7.28), and others by means of solution of a corresponding number of remaining equations of system (7.28).

Let us turn to the question of appraisal of asymptotic behavior of the norm of solution of a system of equations relative to canonical components. The simplest appraisal of asymptotic behavior of the norm of its solution is connected with inequality (3.46)

$$\exp\int_{t_0}^{t}\mu_1\,dt \leqslant \frac{r(t)}{r(t_0)} \leqslant \exp\int_{t_0}^{t}\mu_n\,dt,$$

necessary explanations to which are given in § 1, Chapter IV. Because of this inequality, norm of solution is a bounded function of time if the same function is magnitude

$$\int_{t_0}^{t}\mu_n\,dt,$$

and vanishing function of time if shown magnitude is a vanishing function.

Since coefficients of equations (4.1) in this case are periodic of period Ω, then functions $\mu_1(t)$ and $\mu_n(t)$, which they determine, also are periodic of the same period. Because of this

$$\int_{t}^{t+k\Omega}\mu_n\,dt = k\int_{0}^{\Omega}\mu_n\,dt, \tag{7.29}$$

where k is any natural number, and, consequently, norm of solution is a bounded function of time, if

$$\int_{0}^{\Omega}\mu_n\,dt \leqslant 0, \tag{7.30}$$

and a vanishing function of time, if

$$\int_{0}^{\Omega}\mu_n\,dt < 0. \tag{7.31}$$

Right side of inequality (3.46), taking into account equation (7.29), may be presented during $t = t_0 + \Omega$ in the form

$$\frac{r(t_0+\Omega)}{r(t_0)} \leqslant \exp\int_{0}^{\Omega}\mu_n\,dt. \tag{7.32}$$

In order to strengthen appraisal (7.32) and, correspondingly, to weaken sufficient conditions of limitedness of norm of solution (7.30) and its convergence

to zero (7.31), we will introduce, during determination of extrema of function $\dot{r}/r$ on sphere $\sum_{i=1}^{n} e_i\bar{e}_i = 1$ additional conditions considering periodic properties of squares of moduli of phase coefficients or logarithmic derivative of norm of solution. In accordance with that presented in the preceding section, these magnitudes not always are periodic if among the characteristic indices there are no multiples. Subsequently, it is assumed that this requirement is satisfied.

As the first variant of a more precise definition of appraisals (7.32) we will consider determination of the upper boundary of possible values of magnitude $r(\Omega)/r(0)$ during observance of additional conditions

$$\int_0^{\Omega} D_i\,dt = 0 \quad (i = 1, \ldots, n), \tag{7.33}$$

where $D_i = D_i(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)$ $(i = 1, \ldots, n)$ are Hermitian forms from variables $e_1, \ldots, e_n$, found as derivatives of forms $e_i\bar{e}_i$ during change of variables $e_1, \ldots, e_n$ according to system of equations (3.13).

We will determine these forms.

Carrying out differentiation of Hermitian forms $e_i\bar{e}_i$ $(i = 1, \ldots, n)$, we will find

$$\frac{d}{dt}(e_i\bar{e}_i) = \dot{e}_i\bar{e}_i + e_i\dot{\bar{e}}_i \quad (i = 1, \ldots, n). \tag{7.34}$$

After expressing derivatives of phase coefficients through magnitudes of the coefficients themselves according to equations (3.13) and, analogously, magnitudes conjugate with derivatives of phase coefficients through magnitudes conjugate with phase coefficients, we will obtain

$$\frac{d}{dt}(e_i\bar{e}_i) = \left\{\begin{array}{l} = [(\rho_i + \bar{\rho}_i) - 2G]\,e_i\bar{e}_i + \bar{e}_i\sum\limits_{j=1}^{n} g_{ij}e_j + e_i\sum\limits_{j=1}^{n} \bar{g}_{ij}\bar{e}_j = \\ = [\zeta_i^{(l)} + \bar{\zeta}_i^{(l)} - 2G]\,e_i\bar{e}_i + \bar{e}_i\sum\limits_{j=1}^{n} h_{ij}^{(l+1)}e_j + e_i\sum\limits_{j=1}^{n} \bar{h}_{ij}^{(l+1)}\bar{e}_j = \end{array}\right\} = H_i - 2Ge_i\bar{e}_i \quad (i = 1, \ldots, n), \tag{7.35}$$

where

$$G = G(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n) =$$

$$= \left\{\begin{array}{l} = \frac{1}{2}\sum\limits_{j=1}^{n}\left[(\rho_j + \bar{\rho}_j)\,e_j\bar{e}_j + \bar{e}_j\sum\limits_{k=1}^{n} g_{jk}e_k + e_j\sum\limits_{k=1}^{n} \bar{g}_{jk}\bar{e}_k\right] \\ = \frac{1}{2}\sum\limits_{j=1}^{n}\left[(\zeta_j^{(l)} + \bar{\zeta}_j^{(l)})\,e_j\bar{e}_j + \bar{e}_j\sum\limits_{k=1}^{n} h_{jk}^{(l+1)}e_k + e_j\sum\limits_{k=1}^{n} \bar{h}_{jk}^{(l+1)}\bar{e}_k\right] \end{array}\right.$$

$$H_i = \left\{\begin{array}{l} = (\rho_i + \bar{\rho}_i)\,e_i\bar{e}_i + \bar{e}_i\sum\limits_{j=1}^{n} g_{ij}e_j + e_i\sum\limits_{j=1}^{n} \bar{g}_{ij}\bar{e}_j, \\ = (\zeta_i^{(l)} + \bar{\zeta}_i^{(l)})\,e_i\bar{e}_i + \bar{e}_i\sum\limits_{j=1}^{n} h_{ij}^{(l+1)}e_j + e_i\sum\limits_{j=1}^{n} \bar{h}_{ij}^{(l+1)}\bar{e}_j. \end{array}\right.$$

Let us assume that $i = 1, 2, \ldots, n - 2m$ are indices of real roots λ_i or $\zeta_i^{(l)}$, $n - 2m + 1$ and $n - m + 1$, $n - 2m + 2$ and $n - m + 2$, ..., $n - m$ and n are indices of conjugate complex roots λ_i and $\zeta_i^{(l)}$. Then, as is easy to perceive from the formula for Hermitian form H_i during $i \leq n - m$, all H_i are different, and during $i > n - 2m$

$$H_i = H_{m+i}. \tag{7.36}$$

Since the region of possible values of magnitude $r(\Omega)/r(0)$, determined with the help of equality

$$\frac{r(\Omega)}{r(0)} = \exp \int_0^\Omega G(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)\, dt$$

under the condition that phase coefficients satisfy system of equations (3.13) and condition (3.6), is included in the region of possible values of magnitude $\int_0^\Omega G(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)\, dt$ during arbitrary change of phase coefficients satisfying only certain conditions considering periodicity of those or other magnitudes and condition (3.6), then, it is obvious that, as the upper boundary of the first region there may be selected the exact boundary of the second region. Considering condition of periodicity (7.33) and property (7.36), we will formulate condition for determination of region of possible values of magnitude $r(\Omega)/r(0)$ in the following form

$$\left.\begin{array}{l} \int_0^\Omega G\, dt = \max; \\ \int_0^\Omega (H_i - 2G\bar{e}_i e_i)\, dt = 0 \\ (i = 1, \ldots, n - m). \end{array}\right| \tag{7.37}$$

For solution of system of equations (7.37) we will produce replacement of variables $e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n$ by variables $\underline{f}_1, \ldots, f_{\underline{n}}$ by formulas (3.53). Then Hermitian form $G(e_1, \ldots, e_n, e_1, \ldots, e_n)$ will be turned into quadratic form $F(\underline{f}_1, \ldots, f_n)$ (see § 9, Chapter III), Hermitian forms $H_i(e_1, \ldots, e_{\underline{n}}, \bar{e}_1, \ldots, \bar{e}_n)$ into certain quadratic forms $K_i(f_1, \ldots, f_n)$, and $e_i e_i$ either into f_i^2 if phase coefficient e_i is real or into $1/2\,(f_i^2 + f_{i+m}^2)$ if phase coefficient e_i is complex, $f_i = \sqrt{2}\,\mathrm{Re}\, e_i$, $f_{i+m} = \sqrt{2}\,\mathrm{Im}\, e_i$.

Conditions (7.37) take the form

$$
\left.
\begin{aligned}
&\int_0^\vartheta F\,dt = \max, \\
&\int_0^\vartheta (K_i - 2F f_i^2)\,dt = 0 \qquad (i = 1, \ldots, n-2m), \\
&\int_0^\vartheta [K_i - F(f_i^2 + f_{i+m}^2)]\,dt = 0 \\
&\qquad\qquad (i = n-2m+1, \ldots, n-m).
\end{aligned}
\right\} \qquad (7.38)
$$

Problem of determining functions $f_1(t), \ldots, f_n(t)$, satisfying system of equations (7.38), pertains to a class of isoperimetric problems of the calculus of variation [42]. It is resolved by Euler's method in the following manner.

Introducing n - m unknown constants $\nu_1, \ldots, \nu_{n-m}$, we replace condition (7.38) by equivalent conditions

$$
\left.
\begin{aligned}
&\int_0^\vartheta \{F + \nu_1(K_1 - 2G f_1^2) + \ldots + \nu_{n-2m}(K_{n-2m} - 2G f_{n-2m}^2) + \\
&\quad + \nu_{n-2m+1}[K_{n-2m+1} - G(f_{n-2m+1}^2 + f_{n-m+1}^2)] + \ldots \\
&\quad \ldots + \nu_{n-m}[K_{n-m} - G(f_{n-m}^2 + f_n^2)]\}\,dt = \max, \\
&\int_0^\vartheta (K_i - 2G f_i^2)\,dt = 0 \qquad (i = 1, \ldots, n-2m), \\
&\int_0^\vartheta [K_i - G(f_i^2 + f_{i+m}^2)]\,dt = 0 \qquad (i = n-2m+1, \ldots, n-m)
\end{aligned}
\right\} \qquad (7.39)
$$

and constitute system of equations

$$
\left.
\begin{aligned}
&\nu(t) f_i + \frac{\partial P}{\partial f_i} = 0 \qquad (i = 1, \ldots, n), \\
&\sum_{i=1}^{n} f_i^2 = 1,
\end{aligned}
\right\} \qquad (7.40)
$$

where symbol P is designated integrand of first integral of system (7.39), $\nu(t)$ is one more unknown function.

First n equations of system (7.40) are the usual system of equations determining extremal [42]; the last equation is considered an always considerable, in similar cases (see Chapter IV), property of phase coefficients.

From system of equations (7.40) we determine magnitudes $f_1, \ldots, f_n, \nu$ as functions of argument t and parameters $\nu_1, \ldots, \nu_{n-m}$. Using the found dependence, it is possible to represent polynomials $K_i(f_1, \ldots, f_n) - 2f_i^2 F(f_1, \ldots, f_n)$ $(i = 1, \ldots, n - 2m)$ and $K_i(f_1, \ldots, f_n) - (f_i^2 + f_{i+m}^2)F(f_1, \ldots, f_n)$ $(i = n - 2m + 1, \ldots, n - m)$ in the form of functions $L_i(t, \nu_1, \ldots, \nu_{n-m})$ $(i = 1, \ldots, n - m)$: where t is the argument and $\nu_1, \ldots, \nu_{n-m}$ are constant parameters

and find value of magnitudes $\nu_1, \ldots, \nu_{n-m}$ as solution of system of integral equations

$$\int_0^\Omega L_i\,dt = 0 \quad (i = 1, \ldots, n-m). \tag{7.41}$$

It is necessary to note that system (7.40) has several solutions; however, we are interested only in that which leads to maximum value of magnitude $r(\Omega)/r(0)$. This solution should be either selected from all solutions according to any additional considerations before transition to system (7.41) or determined by means of comparison of all solutions of systems (7.40)-(7.41) after substitution of them in $F(f_1, \ldots, f_n)$.

Appraisal of norm of solution r by the presented method, in principle, should definitize appraisal (7.32) in all cases besides case n = 2 during complex conjugate roots λ_1 and λ_2 or $\zeta_1^{(l)}$ and $\zeta_2^{(l)}$.

In the latter case

$$e_1\bar{e}_1 = e_2\bar{e}_2 = \frac{1}{2}(e_1\bar{e}_1 + e_2\bar{e}_2) = \frac{1}{2}$$

and forms $H_1 - 2Ge_1\bar{e}_1$ and $H_2 - 2Ge_2\bar{e}_2$ turn identically into zero.

In an analogous way it is possible to construct further, more precise definitions of appraisals (7.32), considering, for instance, periodicity of derivatives of squares of moduli of phase coefficients if coefficients of equation (0.1) are a corresponding number of times differentiable. On the other hand, there can be obtained less precise definitions of appraisals (7.32) without using conditions of periodicity of certain forms H_i.

We will consider now variants of more precise definition of appraisals (7.32) by means of calculating periodic properties of logarithmic derivative of norm of solution r. These variants, in all their varieties, require that coefficients of a system of equations relative to canonical components be, a corresponding number of times, differentiable.

Let us assume that coefficients of a system of equations relative to canonical components are at least once differentiable. Then for derivative of logarithmic derivative of norm of its solution we will obtain

$$\frac{d}{dt}\left(\frac{\dot r}{r}\right)=\frac{\partial G}{\partial t}+\sum_{i=1}^{n}\dot e_i\frac{\partial G}{\partial e_i}+\sum_{i=1}^{n}\dot{\bar e}_i\frac{\partial G}{\partial \bar e_i}=$$

$$=\frac{\partial G}{\partial t}+\left\{\begin{aligned}&+\sum_{i=1}^{n}\left[\frac{\partial G}{\partial e_i}\sum_{j=1}^{n}(\lambda_i\delta_{ij}+g_{ij})e_j+\frac{\partial G}{\partial \bar e_i}\sum_{j=1}^{n}(\bar\lambda_i\delta_{ij}+\bar g_{ij})\bar e_j-\right.\\&\qquad\qquad\left.-G\left(\frac{\partial G}{\partial e_i}e_i+\frac{\partial G}{\partial \bar e_i}\bar e_i\right)\right].\\&+\sum_{i=1}^{n}\left[\frac{\partial G}{\partial e_i}\sum_{j=1}^{n}(\zeta_i^{(l)}\delta_{ij}+h_{ij}^{(l+1)})e_j+\frac{\partial G}{\partial \bar e_i}\sum_{j=1}^{n}(\bar\zeta_i^{(l)}\delta_{ij}+\bar h_{ij}^{(l+1)})\bar e_j-\right.\\&\qquad\qquad\left.-G\left(\frac{\partial G}{\partial e_i}e_i+\frac{\partial G}{\partial \bar e_i}\bar e_i\right)\right].\end{aligned}\right. \tag{7.42}$$

Taking into account that

$$\sum_{i=1}^{n}\left(\frac{\partial G}{\partial e_i}e_i+\frac{\partial G}{\partial \bar e_i}\bar e_i\right)=2G,$$

we will copy equation (7.42) in the form

$$\frac{d}{dt}\left(\frac{\dot r}{r}\right)=M-2G^2, \tag{7.43}$$

where

$$M=\frac{\partial G}{\partial t}+$$

$$+\left\{\begin{aligned}&+\sum_{i=1}^{n}\left[\frac{\partial G}{\partial e_i}\sum_{j=1}^{n}(\lambda_i\delta_{ij}+g_{ij})e_j+\frac{\partial G}{\partial \bar e_i}\sum_{j=1}^{n}(\bar\lambda_i\delta_{ij}+\bar g_{ij})\bar e_j\right].\\&+\sum_{i=1}^{n}\left[\frac{\partial G}{\partial e_i}\sum_{j=1}^{n}(\zeta_i^{(l)}\delta_{ij}+h_{ij}^{(l+1)})e_j+\frac{\partial G}{\partial \bar e_i}\sum_{j=1}^{n}(\bar\zeta_i^{(l)}\delta_{ij}+\bar h_{ij}^{(l+1)})\bar e_j\right].\end{aligned}\right. \tag{7.44}$$

By analogy with above-considered case, conditions for determination of upper boundary of region of possible values of magnitude $r(\Omega)/r(0)$, we will register in the form

$$\left.\begin{aligned}&\int_0^{\Omega}G\,dt=\max,\\&\int_0^{\Omega}(M-2G^2)\,dt=0.\end{aligned}\right\} \tag{7.45}$$

System of equations (7.45) may be solved by the same method which was applied for solution of system (7.37).

Other variants of more precise definition of appraisal (7.32), by means of calculation of periodic properties of logarithmic derivative of norm of oscillations, it is possible to connect with subsidiary conditions

$$\int_0^{\Omega}N_2\,dt=0,\ \int_0^{\Omega}N_3\,dt=0\quad \text{etc.}$$

where $N_i = N_i(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)$ is a certain polynomial from variables $e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n$, obtained from Hermitian form $G(e_1, \ldots, e_n, \bar{e}_1, \ldots, \bar{e}_n)$ by means of its i-multiple differentiation and exclusion of derivatives $\dot{e}_1, \ldots, \dot{e}_n, \dot{\bar{e}}_1, \ldots, \dot{\bar{e}}_n$ (after each differentiation) according to equations (3.13). Each such condition considers periodicity of corresponding derivative logarithmic derivative of norm of oscillations.

Condition

$$\int_0^\omega N_i\,dt = 0 \tag{7.46}$$

it is possible to apply if coefficients of system of equations relative to canonical components are i times differentiable.

More precise definition of appraisal (7.32) during application of shown additional conditions (one or simultaneously several), can be obtained by a method analogous to that considered above.

We will designate by symbols e_i' $(i = 1, \ldots, n)$ phase coefficients $e_i(t)$ found as a result of solution of the problem on hand to conditional extremum (maximum). Then, obviously,

$$G(e_1', \ldots, e_n', \bar{e}_1', \ldots, \bar{e}_n') \leq \mu_n. \tag{7.47}$$

If one were to designate left part of inequality (7.47) by the symbol μ_n', then it is possible to record obtained result in the form of inequality

$$\frac{r(\Omega)}{r(0)} \leq \exp\int_0^\Omega \mu_n'\,dt. \tag{7.48}$$

In accordance with inequality (7.48) the new, weakened, sufficient condition of limitedness of norm of solution r we will register in the form

$$\int_0^\Omega \mu_n'\,dt \leq 0, \tag{7.49}$$

and sufficient condition of its convergence to zero — in the form

$$\int_0^\Omega \mu_n'\,dt < 0. \tag{7.50}$$

Thus, it is established how, using specific character of periodically variable coefficients of equation of free oscillations, to obtain more effective criteria for identification of asymptotic properties of norm of solution r. Because of definition of stability, given in § 1 Chapter V, and inequality (4.7)

$$|x|^2 \leq nr^2$$

by revealed asymptotic properties of norm of solution, it is simple to establish the presence of stability or asymptotic stability of oscillations. Namely: for stability of oscillations it is sufficient that there be executed inequality (7.49); for asymptotic stability it is sufficient that there be executed inequality (7.50). This is true also during multiple indices.

Example: We will define residual conditions of stability and asymptotic stability of oscillations, presented by equation

$$\ddot{x} + a\dot{x} + b(1 - \varepsilon \sin t)x = 0$$
$$a > 0, \quad b > 0, \quad \varepsilon > 0. \tag{7.51}$$

In the beginning we will investigate a case of small values of coefficient a. During

$$a^2 < 4b(1 - \varepsilon) \tag{7.52}$$

roots of equation

$$\lambda^2 + a\lambda + b(1 - \varepsilon \sin t) = 0$$

are different during all t and have the form

$$\lambda_{1,2} = -\frac{a}{2} \pm i\sqrt{b(1-\varepsilon) - \frac{a^2}{4}}.$$

After expanding solution of equation (7.51) into canonical components y_1 and y_2 by the method given in § 3 Chapter II, we will come to a system of equations (2.34) in which

$$g_{11} = g_{22} = -g_{12} = -g_{21} = \frac{b\varepsilon \cos t}{4b(1 - \varepsilon \sin t) - a^2}.$$

Designating

$$\frac{b\varepsilon \cos t}{4b(1 - \varepsilon \sin t) - a^2} = g,$$

we will obtain equation (3.43)

$$\det\left\| \frac{g_{jk} + \bar{g}_{kj}}{2} + \delta_{jk}\left(\frac{\lambda_j + \bar{\lambda}_j}{2} - \mu\right)\right\|_1^n = 0$$

in the form

$$\mu^2 - 2\mu\left(-\frac{a}{2} + g\right) + \frac{a^2}{4} - ag = 0.$$

The biggest root of this equation has the form

$$\mu_2 = -\frac{a}{2} + \max(0, g) = -\frac{a}{2} + \max\left(0, \frac{b\varepsilon \cos t}{4b(1 - \varepsilon \sin t) - a^2}\right) =$$
$$= -\frac{a}{2} + \max\left\{0, -\frac{1}{2}\ln\left[4b(1 - \varepsilon \sin t) - \frac{a^2}{4}\right]\right\}.$$

Let us note that

$$\frac{\varepsilon \cos t}{4b(1 - \varepsilon \sin t) - a^2} > 0 \quad \text{for} \quad -\frac{\pi}{2} < t < \frac{\pi}{2},$$
$$\frac{\varepsilon \cos t}{4b(1 - \varepsilon \sin t) - a^2} < 0 \quad \text{for} \quad \frac{\pi}{2} < t < \frac{3\pi}{2}.$$

Therefore

$$\frac{1}{\Omega}\int_0^{\Omega}\mu_2\,dt=\frac{1}{2\pi}\int_0^{2\pi}\mu_2\,dt=-\frac{a}{2}-\frac{1}{4\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\ln\left[4b(1-\varepsilon\sin t)-a^2\right]dt=$$

$$=-\frac{a}{2}+\frac{1}{4\pi}\ln\frac{4b(1+\varepsilon)-a^2}{4b(1-\varepsilon)-a^2}.$$

It follows from this that

$$\frac{1}{\Omega}\int_0^{\Omega}\mu_2\,dt<0 \quad\text{or}\quad \frac{1}{\Omega}\int_0^{\Omega}\mu_2\,dt<0,$$

if, correspondingly,

$$\varepsilon<\left(1-\frac{a^2}{4b}\right)\operatorname{th}\pi a \tag{7.53}$$

or

$$\varepsilon>\left(1-\frac{a^2}{4b}\right)\operatorname{th}\pi a. \tag{7.54}$$

Condition (7.53) is sufficient condition of stability; condition (7.54) is sufficient condition of asymptotic stability.

Now we will apply expansion of solution of equation (7.51) into components z_1 and z_2, determining functions $\zeta_1(t)$ and $\zeta_2(t)$ from conditions (7.26), assuming the simplest case, k = 0.

Condition

$$z_{0l}=0$$

leads to equation

$$\zeta^2+a\zeta+b=0,$$

whence

$$\zeta_{1,2}=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}-b}.$$

Designating

$$\zeta_{1,2}=\zeta_{1,2}^{(0)},$$

will find coefficients $h_{ij}^{(1)}$ of system

$$\left.\begin{aligned} z_1&=(\zeta_1^{(0)}+h_{11}^{(1)})z_1+h_{12}^{(1)}z_2,\\ z_2&=h_{21}^{(1)}z_1+(\zeta_2^{(0)}+h_{22}^{(1)})z_2\end{aligned}\right\}$$

in the form

$$h_{11}^{(1)}=h_{12}^{(1)}=\frac{-b\varepsilon\sin t}{\sqrt{a^2-4b}}=-h_{21}^{(1)}=-h_{22}^{(1)}.$$

Assuming that

$$a^2<4b \tag{7.55}$$

we will obtain equation for determination of functions $\mu_1(t)$ and $\mu_2(t)$ in the form

$$\mu^2 + a\mu + \frac{a^2}{4} - \left(\frac{b\varepsilon \sin t}{\sqrt{4b - a^2}}\right)^2 = 0.$$

Solving it, we will find

$$\mu_{1,2} = -\frac{a}{2} \mp \left|\frac{b\varepsilon \sin t}{\sqrt{4b - a^2}}\right|.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} \mu_2\, dt = -\frac{a}{2} + \frac{b\varepsilon}{2\pi\sqrt{4b - a^2}}\int_0^{2\pi} |\sin t|\, dt = -\frac{a}{2} + \frac{4b\varepsilon}{2\pi\sqrt{4b - a^2}}.$$

Consequently, for stability of oscillations it is sufficient that there be executed condition

$$a > \frac{4b\varepsilon}{\pi\sqrt{4b - a^2}}. \tag{7.56}$$

For asymptotic stability it is sufficient to fulfill condition

$$a > \frac{4b\varepsilon}{\pi\sqrt{4b - a^2}}. \tag{7.57}$$

Now we will investigate stability of oscillations (7.51) in the range of values of parameters a, b and ε, at which roots $\lambda_1(t)$ and $\lambda_2(t)$ or $\zeta_1^{(0)}(t)$ and $\zeta_2^{(0)}(t)$ are multiple or close to multiple.

According to method of construction of modified canonical expansion of equation of oscillations, presented in § 5, Chapter II, modification of both above-considered expansions leads to the following conditions of expansion:

$$\left.\begin{aligned} x &= y_1 + y_2, \\ \dot{x} &= -\frac{a}{2} y_1. \end{aligned}\right\}$$

System of equations relative to canonical components y_1 and y_2 we obtain in the form

$$\left.\begin{aligned} \dot{y}_1 &= \left(-\frac{a}{2} + \frac{4b(1 - \varepsilon \sin t) - a^2}{2a}\right) y_1 + \frac{2b(1 - \varepsilon \sin t)}{a} y_2, \\ \dot{y}_2 &= -\frac{4b(1 - \varepsilon \sin t) - a^2}{2a} y_1 - \frac{2b(1 - \varepsilon \sin t)}{a} y_2. \end{aligned}\right\}$$

This system corresponds to form G with matrix of coefficients

$$\begin{Vmatrix} -a + \dfrac{2b(1 - \varepsilon \sin t)}{a} & \dfrac{a}{4} \\ \dfrac{a}{4} & -\dfrac{2b(1 - \varepsilon \sin t)}{a} \end{Vmatrix}.$$

Equation for determination of functions $\mu_1(t)$ and $\mu_2(t)$ we obtain in the form

$$\mu^2 + a\mu - \frac{a^2}{16} + \frac{2b(1 - \varepsilon \sin t)}{1} - \frac{4b^2(1 - \varepsilon \sin t)^2}{a^2} = 0.$$

From this equation we determine

$$\mu_2 = -\frac{a}{2} + \sqrt{\frac{5a^2}{16} - \frac{2b(1-\varepsilon\sin t)}{1} + \frac{4b^2(1-\varepsilon\sin t)^2}{a^2}}.$$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi}\mu_2\,dt = -\frac{a}{2} + \frac{1}{2\pi}\int_0^{2\pi}\sqrt{\frac{5a^2}{16} - 2b(1-\varepsilon\sin t) + \frac{4b^2(1-\varepsilon\sin t)^2}{a^2}}\,dt.$$

Applying known inequality

$$\left[\frac{1}{\Omega}\int_0^{\Omega} f(u)\,du\right]^2 \leq \frac{1}{\Omega}\int_0^{\Omega}[f(u)]^2\,du,$$

from preceding equation we will obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\mu_2\,dt < -\frac{a}{2} + \sqrt{\frac{5a^2}{16} - 2b + \frac{4b^2}{a^2}\left(1+\frac{\varepsilon^2}{2}\right)}.$$

It follows from this that for stability of oscillations it is sufficient that there be executed condition

$$\frac{a^2}{4} > \frac{5}{16}a^2 + \frac{4b^2}{a^2} - 2b + \frac{2b^2}{a^2}\varepsilon^2;$$

for asymptotic stability this condition should be executed with the sign of absolute inequality.

Solving this inequality relative to ε, we will obtain condition of stability in the form

$$\varepsilon \leq \frac{a}{b}\sqrt{b - \frac{2b^2}{a^2} - \frac{a^2}{32}} \qquad (7.58)$$

and asymptotic stability in the form

$$\varepsilon < \frac{a}{b}\sqrt{b - \frac{2b^2}{a^2} - \frac{a^2}{32}}. \qquad (7.59)$$

It remained for us to investigate stability of oscillations during large values of a. We will use for this the earlier expansion of solution of equation (7.51) to components z_1 and z_2.

Replacing condition (7.55) by condition

$$a^2 > 4b, \qquad (7.60)$$

we will obtain an equation for determination of functions $\mu_1(t)$ and $\mu_2(t)$ in the form

$$\left(\mu + \frac{a}{2} + \sqrt{\frac{a^2}{4} - b} + \frac{b\varepsilon\sin t}{\sqrt{a^2-4b}}\right)\times$$
$$\times\left(\mu + \frac{a}{2} - \sqrt{\frac{a^2}{4} - b} - \frac{b\varepsilon\sin t}{\sqrt{a^2-4b}}\right) = 0.$$

Solving it, we will find

$$\mu_{1,2} = -\frac{a}{2} \mp \sqrt{\frac{a^2}{4} - b} \mp \frac{b\varepsilon \sin t}{\sqrt{a^2 - 4b}}.$$

During

$$\varepsilon < \frac{a^2 - 4b}{2b} \tag{7.61}$$

we will obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \mu_2 \, dt = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b}. \tag{7.62}$$

If condition (7.61) is not executed, then

$$\frac{1}{2\pi}\int_0^{2\pi} \mu_2 \, dt = -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b} + \frac{2b\varepsilon}{\sqrt{a^2 - 4b^2}} \int_\alpha^\beta |\sin t| \, dt, \tag{7.63}$$

where α and β are boundaries of interval of those values of t for which

$$a^2 - 4b - 2b\varepsilon \sin t < 0.$$

Because of equalities (7.62), for stability and asymptotic stability of oscillations it is sufficient that there be executed conditions (7.60) and (7.61).

Equality (7.63) it is possible to replace by inequality

$$\frac{1}{2\pi}\int_0^{2\pi} \mu_2 \, dt \leq -\frac{a}{2} + \sqrt{\frac{a^2}{4} - b} + \frac{2b\varepsilon}{\sqrt{a^2 - 4b}} \int_0^{2\pi} |\sin t| \, dt.$$

There follows from this condition of stability

$$\varepsilon \leq \frac{\pi}{8}\left(a^2\sqrt{a^2 - 4b} - a^2 + 4b\right) \tag{7.64}$$

and asymptotic stability

$$\varepsilon < \frac{\pi}{8}\left(a^2\sqrt{a^2 - 4b} - a^2 + 4b\right). \tag{7.65}$$

Graphs of dependence $\varepsilon(a)$ during $b = 1$, built on conditions (7.53), (7.56), (7.58), and (7.61) during observance of sign of equality, are presented on Fig. 1[illegible]. By a solid line with shading is shown boundary of region of stability, guaranteed by all obtained conditions of stability in totality. It is obvious that inside this region stability is asymptotic.

Equation (7.51) was investigated in works [43-47]. We will compare our results with sufficient conditions of stability obtained in work [47], the latter of the mentioned works.

For case $b = 1$ these conditions have the form

$$a < \sqrt{1 + \varepsilon} - \sqrt{1 - \varepsilon}, \tag{7.66}$$

$$a < \frac{\varepsilon}{2\sqrt{1 - \varepsilon^2}}. \tag{7.67}$$

where stability in the first case is asymptotic

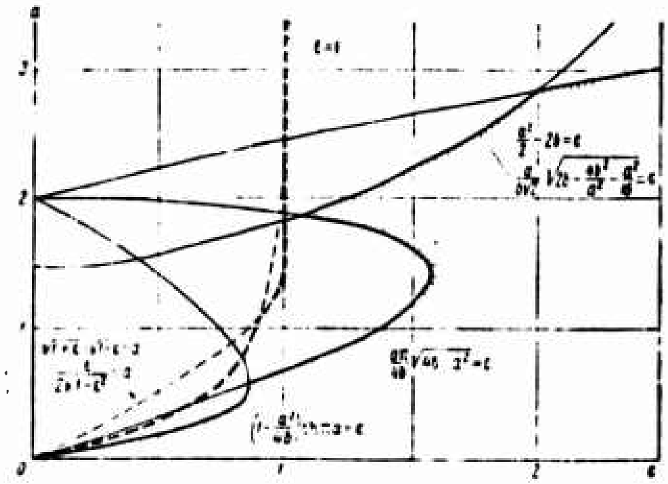

Fig. 13. Region of stability of oscillations, presented by equation $\ddot{x}+a\dot{x}+b(1+\varepsilon\sin t)x=0$ during b = 1 (shading shows boundary of stability).

Boundaries of region of stability, delivered by conditions (7.66) and (7.67), are presented on Fig. 13 by hachured lines. By hachured line with shading is shown boundary of region of stability guaranteed by both conditions in totality. From the graphs it is seen that the region of stability determined in work (47) lies completely within the region of stability determined according to the conditions found above.

CHAPTER VIII

FREE OSCILLATIONS, REPRESENTED BY EQUATIONS WITH EXPONENTIAL AND COMPOUND-EXPONENTIAL COEFFICIENTS

§ 1. Equations of Free Oscillations with Exponential and Compound-Exponential Coefficients

Let us assume that in equation (0.1)

$$\frac{d^n x}{dt^n} + b_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + b_n x = 0$$

coefficients $b_1, \ldots, b_n$ are exponential functions of time, i.e., have the form

$$b_i = B_i t^{\beta_i} \qquad (i = 1, \ldots, n), \tag{8.1}$$

where B_i and β_i ($i = 1, \ldots, n$) are real numbers, and t^{β_i} for every value of t and i are principal values of degree,[1] and at least for one value of i $B_i \neq 0$ and $\beta_i \neq 0$.

Then equation (0.1) we will call an equation with exponential coefficients.

Obviously, particular form of exponential coefficients consists of constant

---

[1]Exponential function t^{β_j} in general is a many-valued function. Its general determination has the form

$$t^{\beta_j} = e^{\beta_j(\ln t + 2k\pi i)} \quad (i = \sqrt{-1}, \ k \text{ — is an integer}).$$

Principal value of function is obtained from this formula during $k = 0$, i.e., is determined by equality

$$t^{\beta_j} = e^{\beta_j \ln t}.$$

Thus, for instance, if $\beta_j = 1/2$, then for principal value we will obtain

$$t^{\beta_j} = +\sqrt{t}.$$

coefficients. Consequently, some of the coefficients of an equation with exponential coefficients can be constants.

Exponential coefficients $B_i t^{\beta_i}$ $(i = 1, \ldots, n)$ in interval $(0, \infty)$ are (any amount of times) differentiable functions, expandable, in environment

$$|t-t_0|<t_0$$

of any point t_0 belonging to this interval, to convergent power series

$$B_i t_0^{\beta_i}\left[1+\frac{(t-t_0)\beta_i}{t_0}+\frac{(t-t_0)^2\beta_i(\beta_i-1)}{t_0^2\cdot 1\cdot 2}+\ldots\right] \qquad (i=1,\ldots,n).$$

and consequently, constitute functions, analytical in interval $(0, \infty)$. They are limited in any finite interval belonging to interval $(0, \infty)$ and do not turn into zero at any point of last interval, if $B_i \neq 0$; if $B_i = 0$, then at $0 < t < \infty$ value $b_i \equiv 0$.

Coefficients of equation (0.1) we will call compound-exponential if they have the form

$$b_i=\frac{\sum\limits_{j=1}^{l} B_{ij}t^{\beta_{ij}}}{\sum\limits_{j=1}^{m} C_{ij}t^{\gamma_{ij}}}, \tag{8.2}$$

where B_{ij}, β_{ij} $(j = 1, \ldots, l)$, C_{ij}, γ_{ij} $(j = 1, \ldots, m)$ are real numbers, and $t^{\beta_{ij}}$ and $t^{\gamma_{ij}}$ for each value of t and j are principal values of degree but cannot be represented in the form (8.1) (i.e., are not exponential). We will consider that $\beta_{i,j+1} > \beta_{ij}$ and $\gamma_{i,j+1} > \gamma_{ij}$.

Compound-exponential coefficients are functions (any amount of times) differentiable at <u>almost all</u> points of interval $(0, \infty)$; they are expanded in quite small environments of these nonsingular points into convergent power series and, thus, constitute functions which are analytical almost everywhere in interval $(0, \infty)$. They can lose the properties of analytical functions at those points where magnitudes

$$\sum_{j=1}^{m} C_{ij}t^{\gamma_{ij}}$$

turn into zero.

Since equation

$$\sum_{j=1}^{m} C_{ij}t^{\gamma_{ij}}=0 \tag{8.3}$$

has a limited number of roots, then it is always possible to indicate such $t = T$ at which all limited, real, positive roots of equation (8.3) belong to interval $(0, T)$. Moreover in interval (T, ∞) function $b_i(t)$ is analytical, where it is limited in

any finite interval belonging to interval (T, ∞).

Compound-exponential coefficient can turn into zero at certain points of interval $(0, \infty)$. However, it is always possible to indicate such $t = T_1$ at which all finite positive values of t, turning it into zero, belong to interval $(0, T_1)$. If magnitude T_1 is no larger than magnitude T, coefficient $b_i(t)$ does not turn into zero at any point of interval (T, ∞).

If during selection of magnitude T there are observed conditions shown in the last two paragraphs, then in interval (T, ∞) compound-exponential coefficient is an analytical function and does not take zero values. Using idea of asymptotic equivalence (in the sense that $x(t)$ and $y(t)$ are asymptotically equivalent if

$$\lim_{t \to \infty} \frac{y(t)}{x(t)} = 1\Big),$$

we will note the following property of compound-exponential coefficients: compound-exponential coefficient (8.2) is asymptotically equivalent to exponential coefficient

$$b_i = \frac{B_{il}}{C_{im}} t^{\beta_{il} - \gamma_{im}}. \tag{8.4}$$

This property is easy to reveal, estimating magnitude (8.2) during $t \to \infty$.

It is expedient, however, to note that from the asymptotic equivalence of coefficients, coinciding in indices, of two compared equations of form (0.1), asymptotic equivalence of their solutions by no means follow (see below-mentioned example).

If in equation (0.1) one of the coefficients is compound-exponential and the others are compound-exponential or exponential, then this equation we will call an <u>equation with compound-exponential coefficients</u>.

When determining exponential and compound-exponential coefficients, we did not connect with any limitations indices of degree β_i, β_{ij}, γ_{ij} besides the requirement of their realness. If coefficients of equation (0.1) are exponential or compound-exponential and magnitudes β_i, β_{ij}, γ_{ij} are natural numbers or zeroes, then, leading all coefficients b_i to a common denominator (in case of an equation with exponential coefficients it is not necessary to do this since the denominators of all coefficients in this case are equal to unity), we will obtain equation (0.1) in the form

$$P_0(t) \frac{d^n x}{dt^n} + P_1(t) \frac{d^{n-1} x}{dt^{n-1}} + \ldots + P_n(t) x = 0. \qquad (8.[illegible])$$

where $P_0(t)$, $P_1(t)$, ..., $P_n(t)$ are polynomials from t.

This equation, which, thus, is a particular case of equations of forms considered by us, is called, in the theory of linear differential equations, an <u>equation with</u>

polynomial coefficients. Consequently, equations considered in this chapter are equations of a certain class, more general than the class of equations with polynomial coefficients.

Example. Equation

$$\ddot{x} + t\dot{x} + \left(\frac{t^2}{4} + \frac{1}{2}\right)x = 0 \tag{8.6}$$

is an equation with compound-exponential coefficients. Its general solution, given in work [25], has the form

$$x(t) = e^{-\frac{t^2}{4}}(C_1 + C_2 t).$$

Coefficient

$$b_2 = \frac{t^2}{4} + \frac{1}{2}$$

is asymptotically equivalent to coefficient

$$b_2 = \frac{t^2}{4}.$$

Replacing coefficient b_2 of equation (8.6) by this, its asymptotic equivalent, we will obtain equation

$$\ddot{x} + t\dot{x} + \frac{t^2}{4}x = 0. \tag{8.7}$$

General solution of this equation has the form

$$x(t) = e^{-\frac{t^2}{4}}\left(C_1 \exp\frac{t}{\sqrt{2}} + C_2 \exp\frac{-t}{\sqrt{2}}\right).$$

It is easy to see that not one of the particular solutions of equation (8.7) is an asymptotic equivalent of any particular solution of equation (8.6).

Both equations (8.7) and (8.6) belong to the class of equations with polynomial coefficients.

§ 2. Proposed Form of General Solutions of Equations with Exponential and Compound-Exponential Coefficients

We will assume that during sufficiently large T general solutions of equations of free oscillations with exponential and compound-exponential coefficients in interval (T, ∞) can be represented in the form

$$x(t) = \sum_{i=1}^{n} C_i \exp\int_{T}^{t} \sum_{j=1}^{\infty} H_{ij} t^{\eta_{ij}}\, dt. \tag{8.8}$$

where C_i ($i = 1, \ldots, n$) are complex arbitrary constants; H_{ij} and η_{ij} ($i = 1, \ldots, n$, $j = 1, 2 \ldots$) are certain fixed constants, the first of which is complex, and the second must be real, where connected by relationship $\eta_{i1} > \eta_{i2} > \ldots$.

We will not try to prove the validity of such an assumption, but, using it constructively, we will find solution of certain interesting problems. Correctness

of found solutions we will prove.

Without decreasing generality of analysis, we will consider only such equations for which

$$b_n(t) \neq 0.$$

This is justified by the fact that during nonfulfillment of given condition, substitution

$$x_1 = \dot{x}$$

leads to an equation with an order lowered to unity, whose approximate presentation of general solution during

$$b_{n-1}(t) \neq 0$$

may be determined by the method presented below. If, however, coefficient $b_{n-1}(t)$ is identically equal to zero, then substitution $x_1 = \dot{x}$ it is possible to replace by substitution $x_1 = \ddot{x}$ and to determine approximate presentation of general solution of equation, differing from initial by an order of two. These reasonings, in an obvious manner, spread to cases when there are identically equal to zero other coefficients, preceding in indices coefficient b_{n-1}.

After determining approximate presentation of the general solution of an equation of lowered order, there can be obtained an approximate presentation of general solution of the initial equation by means of integration of the first. Appearing as a result of integration, an arbitrary constant or a polynomial obtained generally from t with arbitrary constants as coefficients, determine the family of nonzero particular solutions of the initial equation, satisfying condition

$$x_1 = 0.$$

Considering that condition $b_n(t) \neq 0$ is executed and, consequently, general solution does not contain components which are constants, we will consider subsequently that $H_{1i} \neq 0$, for all i.

As a result of the integration of integrands in equation (8.8) the latter takes the form

$$x(t) = \sum_{i=1}^{n} C_i' t^{H_i} \exp \sum_{\substack{j=1, \\ j \neq k, \text{ if } \eta_{ik} = -1}}^{\infty} \frac{H_{ij}}{\eta_{ij}+1} t^{\eta_{ij}+1}, \tag{8.9}$$

where C_i' ($i = 1, \dots, n$) are new arbitrary constants; H_i for every $i = 1, \dots, n$ are such magnitudes of H_{ij} ($j = 1, 2, \dots,$) for which $\eta_{ij} = -1$.

Factors t^{H_i} appear at the expense of integration of functions

$$H_{ij} t^{\eta_{ij}} = H_{ij} t^{-1}.$$

since

$$\exp \int_T^t H_{ij} t^{-1} dt = \exp H_{ij} (\ln t - \ln T) = \frac{t^{H_{ij}}}{T^{H_{ij}}}.$$

If series $\sum_{j=1}^{\infty} H_{ij} t^{\eta_{ij}}$ does not contain for a given value of index i exponent $\eta_{ij} = -1$, then in equation (8.9) one should put $H_i = 0$.

Example: Given in the preceding section, the formula of general solution of equation (8.6) is obtained from formula (8.9) if one assumes

$$H_1 = 0, \quad H_{11} = -\frac{1}{4}, \quad \eta_{11} = 1,$$
$$H_{1j} = 0 \text{ when } j > 1; \quad H_2 = 1, \quad H_{21} = -\frac{1}{4}, \quad \eta_{21} = 1,$$
$$H_{2j} = 0 \text{ when } j > 1.$$

General solution of equation (8.7) (see § 1) we will obtain if we assume

$$H_1 = H_2 = 0, \quad H_{11} = H_{21} = -\frac{1}{4}, \quad \eta_{11} = \eta_{21} = 1,$$
$$H_{12} = \frac{1}{\sqrt{2}}, \quad H_{22} = -\frac{1}{\sqrt{2}}, \quad \eta_{12} = \eta_{22} = 0, \quad H_{1j} = H_{2j} = 0$$

during $j > 2$.

In the considered examples under signs of sums enter finite numbers of terms differing from zero, i.e., series are summarized and finite. It would have been possible to present many examples of known solutions, presented in the form (8.9), with infinite summarized series.

§ 3. Method of Determining Coefficients H_{ij} and η_{ij}

As was noted in § 1, coefficients b_i (i = 1, ..., n) of equations with exponential and compound-exponential coefficients are analytic functions of t in interval (T, ∞) if magnitude T is selected sufficiently great. Considering, namely, such selection of T, it is possible to expand solution of equation (0.1) in shown interval into components z_1, ..., z_n applying formula of expansion (2.39) and assigning somehow functions $\zeta_1(t)$, ..., $\zeta_n(t)$. These functions can be given by different methods. One of the methods was presented in Chapter II for an equation of free oscillations with coefficients which are a sufficient number of times differentiable, which, in particular, are analytic functions. It gives the possibility of constructing a sequence of sets of functions $\{\zeta_1(t), ..., \zeta_n(t)\}$ with finite or infinite number of terms, where each of the systems determines corresponding canonical expansion of solution of the considered equation. Here we will consider the second method, using (expressed in the preceding paragraph) the assumption about the form of general solutions of equations, studied in this chapter.

This method also gives the possibility of determining a sequence of sets of functions $\{\zeta_1(t), \ldots, \zeta_n(t)\}$ (consisting of a finite or infinite number of terms), each of whose elements determines a certain canonical expansion of the solution of an equation of oscillations.

Subsequently, considering equation (0.1), we will assume that its coefficients have form (8.2), allowing the possibility of transforming form (8.2) into form (8.1). Thus, our consideration, in an equal measure, will embrace cases of an equation with exponential coefficients and an equation with compound-exponential coefficients.

In order to establish the first element of the unknown sequence, we will turn to general solution of equation (0.1), represented by formula (8.8), and considering $C_i \neq 0$, $C_j = 0$ during $i \neq j$, we will separate particular solutions $x_i(t)$ $(i = 1, \ldots, n)$ in the form

$$x_i(t) = C_i \exp \int_{T}^{t} \sum_{j=1}^{\infty} H_{ij} t^{\gamma_{ij}} dt \qquad (i = 1, \ldots, n). \tag{8.10}$$

Taking into account only higher powers of t, we will replace particular solutions (8.10) by approximate solutions of the form

$$\tilde{x}_i(t) = C_i \exp \int_{T}^{t} H_{i1} t^{\gamma_{i1}} dt \qquad (i = 1, \ldots, n). \tag{8.11}$$

Using conformity between functions $\zeta_i^{(0)}(t)$ $(i = 1, \ldots, n)$ and approximate presentation of general solution of equation (0.1), established by formula (4.20) during $l = 1$ (see § 2 Chapter IV), we will assume, according to equation (8.11), that functions $\zeta_i^{(0)}(t)$ $(i = 1, \ldots, n)$ have the form

$$\zeta_i^{(0)}(t) = H_{i1} t^{\gamma_{i1}} \qquad (i = 1, \ldots, n). \tag{8.12}$$

Formulas (8.11) will obtain the form

$$\tilde{x}_i(t) = C_i \exp \int_{T}^{t} \zeta_i^{(0)}(t)\, dt \qquad (i = 1, \ldots, n). \tag{8.13}$$

Replacing in left part of equation (0.1) unknown solutions $x_i(t)$ $(i = 1, \ldots, n)$ with their approximate presentations $\tilde{x}_i(t)$ $(i = 1, \ldots, n)$, we will obtain, because of formulas (8.13)

$$\frac{d^n\tilde{x}}{dt^n}+b_1\frac{d^{n-1}\tilde{x}}{dt^{n-1}}+\ldots+b_n\tilde{x}=\{(\zeta_i^{(0)}+D)^{n-1}\zeta_i^{(0)}+$$

$$+[(\zeta_i^{(0)}+D)^{n-2}\zeta_i^{(0)}]b_1+\ldots$$

$$\ldots+b_n\}C_i\exp\int_{T}^{t}\zeta_i^{(0)}\,dt \qquad (i=1,\ldots,n). \tag{8.14}$$

Obviously, approximate solution $\tilde{x}_i(t)$ coincides with exact $x_i(t)$, if

$$(\zeta_i^{(0)}+D)^{n-1}\zeta_i^{(0)}+[(\zeta_i^{(0)}+D)^{n-2}\zeta_i^{(0)}]b_1+\ldots+b_n=0. \tag{8.15}$$

While limiting approximate solution by form (8.11), in general we cannot indicate such values of magnitudes H_{i1} and η_{i1} at which is satisfied this condition. However, it is possible to establish such a value for them, at which, in a certain meaning, there is attained maximum accuracy of approximation of the left part of equality (8.15) to zero. These values we will find in the following manner.

Differentiating equalities (8.12) one, two, ..., (n - 1) times we will obtain relationship

$$\left.\begin{aligned}
&\frac{d\zeta_i^{(0)}}{dt}=\eta_{i1}\zeta_i^{(0)}t^{-1},\\
&\frac{d^2\zeta_i^{(0)}}{dt}=\eta_{i1}(\eta_{i1}-1)\zeta_i^{(0)}t^{-2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\frac{d^{n-1}\zeta_i^{(0)}}{dt^{n-1}}=\eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2)t^{-n+1}.
\end{aligned}\right| \tag{8.16}$$

On this basis it is possible to record in the left part of equation (8.15) in the form[1]

$$(\zeta_i^{(0)})^n+c_1^{(i)}(\zeta_i^{(0)})^{n-1}+\ldots+c_n^{(i)}, \tag{8.17}$$

where

$$\left.\begin{aligned}
&c_1^{(i)}=\binom{n}{2}\eta_{i1}t^{-1}+b_1,\\
&c_2^{(i)}=\left[\binom{n}{3}(\eta_{i1}-1)+3\binom{n}{4}\eta_{i1}\right]\eta_{i1}t^{-2}+\binom{n-1}{2}\eta_{i1}b_1t^{-1}+b_2,\\
&c_3^{(i)}=\sum_{j=1}^{n-3}j\eta_{i1}[(\eta_{i1}-1)(\eta_{i1}-2)+3(j-1)(\eta_{i1}-1)+
\end{aligned}\right| \tag{8.18}$$

---

[1]Here and subsequently, symbol $\binom{n}{m}$ designates number of combinations of n elements with respect to m, i.e.,

$$\binom{n}{m}=\frac{n(n-1)(n-2)\ldots(n-m+1)}{1\cdot2\cdot3\cdot\ldots\cdot m}.$$

$$
\begin{aligned}
&+(j-1)(j-2)\tau_{i1}^{2}]t^{-3}+\left[\binom{n-1}{3}(\tau_{i1}-1)+\right.\\
&\left.+3\binom{n-1}{4}\tau_{i1}\right]\tau_{i1}b_1t^{-2}+\binom{n-2}{2}\tau_{i1}b_2t^{-1}+b_3.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&c_{n-1}^{(i)}=\tau_{i1}(\tau_{i1}-1)\ldots(\tau_{i1}-n+2)t^{1-n}+\ldots\\
&\qquad\ldots+\tau_{i1}b_{n-2}t^{-1}+b_{n-1},\\
&c_n^{(i)}=b_n.
\end{aligned}
\qquad (8.18\ \text{Cont'd})
$$

Designating

$$
b_j=b_j^{(i)}t^{-j}\qquad (j=1,\ldots,n). \qquad (8.19)
$$

it is possible to give formulas (8.18) the form

$$
\begin{aligned}
&c_1^{(i)}=\left[\binom{n}{2}\tau_{i1}+b_1^{(i)}\right]t^{-1}=d_1^{(i)}t^{-1},\\
&c_2^{(i)}=\left[\left(\binom{n}{3}(\tau_{i1}-1)+3\binom{n}{4}\tau_{i1}\right]\tau_{i1}+\binom{n-1}{2}\tau_{i1}b_1^{(i)}+\right.\\
&\qquad\left.+b_2^{(i)}\right]t^{-2}=d_2^{(i)}t^{-2},\\
&c_3^{(i)}=\left\{\sum_{j=1}^{n-3}j\tau_{i1}[(\tau_{i1}-1)(\tau_{i1}-2)+3(j-1)(\tau_{i1}-1)+(j-\right.\\
&\quad-1)(j-2)\tau_{i1}^{2}]+\left[\binom{n-1}{3}(\tau_{i1}-1)+3\binom{n-1}{4}\tau_{i1}\right]\tau_{i1}b_1^{(i)}+\\
&\qquad\left.+\binom{n-2}{2}\tau_{i1}b_2^{(i)}+b_3^{(i)}\right\}t^{-3}=d_3^{(i)}t^{-3}.\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&c_{n-1}^{(i)}=[\tau_{i1}(\tau_{i1}-1)\ldots(\tau_{i1}-n+2)+\ldots\\
&\qquad\ldots+\tau_{i1}b_{n-2}^{(i)}+b_{n-1}^{(i)}]t^{1-n}=d_{n-1}^{(i)}t^{1-n},\\
&c_n^{(i)}=b_n^{(i)}t^{-n}=d_n^{(i)}t^{-n}.
\end{aligned}
\qquad (8.20)
$$

Considering these relationships, we will obtain after substitution

$$
\zeta_i^{(0)}=\varkappa_i t^{-1}\qquad (i=1,\ldots,n) \qquad (8.21)
$$

expression

$$
(\zeta_i^{(0)})^n+c_1^{(i)}(\zeta_i^{(0)})^{n-1}+\ldots+c_n^{(i)}=t^{-n}D_i^{(1)}(\varkappa_i). \qquad (8.22)
$$

where

$$
D_i^{(1)}(\varkappa_i)=\varkappa_i^n+d_1^{(i)}\varkappa_i^{n-1}+\ldots+d_n^{(i)}. \qquad (8.23)
$$

Thus, left part of equation (8.15) may be presented in the form (8.22), where coefficients of polynomial $D_i^{(1)}(\varkappa_i)$ are functions of coefficients of equation (0.1) and indices of degree η_{i1} and magnitude $\varkappa_i$ is connected with magnitudes H_{i1} and η_{i1} because of relationships (8.21) and (8.12) by dependence

$$
\varkappa_i=H_{i1}t^{\eta_{i1}+1}. \qquad (8.24)
$$

If one were to express coefficients $d_j^{(i)}$ (j = 1, ..., n) according to formulas

(8.20) through magnitude η_{i1} and coefficients of equation (0.1), after developing the latter in the form of exponential or compound-exponential functions, and magnitude $\varkappa_i$ is presented in the form (8.24) then after reducing to a common denominator (in the case of an equation with exponential coefficients it is not necessary to do this) we will obtain the left part of equation (8.15) in the form of a compound-exponential function, which during certain values of coefficients η_{i1} and H_{i1} in general, can degenerate into exponential. Considering arbitrary values H_{i1} for every value η_{i1} it is possible to present this function in the form

$$t^{-n}D_i^{(1)}(\varkappa_i)=t^{-n}\left[L_{i1}t^{\lambda_{i1}}+o\left(t^{\lambda_{i1}}\right)\right]. \tag{8.25}$$

where magnitude L_{i1} is complex and magnitude λ_{i1} is real.

If $L_{i1}\neq 0$, then in accordance with equality (8.25) during $t\to\infty$ function $t^{-n}D_i^{(1)}(\varkappa_i)$ is asymptotically equivalent to magnitude

$$L_{i1}t^{\lambda_{i1}-n}.$$

Therefore, as a condition of maximum accuracy of the approach of the left part of equality (8.15) to zero, it is logical to take condition

$$L_{i1}=0. \tag{8.26}$$

This condition places in conformity to each value η_{i1} a certain value H_{i1}; it does not put direct limitations on exponent λ_{i1}, with help of which it is possible during $t\to\infty$ to estimate from above numerical growth of magnitude interesting us; however, it is very convenient from the side of construction since it leads to a rather simple method of determining values of coefficients η_{i1} and H_{i1}.

Let us consider at first cases when

$$\lim_{t\to\infty} b_i^{(1)}=0 \qquad (i=1,\ldots,n). \tag{8.27}$$

i.e., when coefficients $b_i^{(1)}$ ($i = 1, \ldots, n$) are vanishing functions of time. Since η_{i1} is a finite number, then coefficients $d_j^{(i)}$ ($j = 1, \ldots, n$) of polynomial $D_i^{(1)}(\varkappa_i)$ in this case are bounded functions.

There can be made three assumptions about the value of magnitude η_{i1}:

$$\eta_{i1}>-1,$$
$$\eta_{i1}=-1,$$
$$\eta_{i1}<-1.$$

In the first case, due to equality (8.24), numerical growth of function $\varkappa_i(t)$ is not limited; member $\varkappa_i^n$ during $t\to\infty$ in sum (8.23) will prevail and, inasmuch as

$H_{i1} \neq 0$, condition (8.26) cannot be carried out.

In the second case magnitude $\varkappa_i$ is constant and, inasmuch as all terms containing coefficients $b_i^{(1)}$ (i = 1, ..., n) are vanishing functions, then during $t \to \infty$ its following part will prevail in sum (8.23):

$$x_i^n + \binom{n}{2}\eta_{i1}x_i^{n-1} + \eta_{i1}\left[\binom{n}{3}(\eta_{i1}-1)+3\binom{n}{4}\eta_{i1}\right]x_i^{n-2} + \ldots$$
$$\ldots + \eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2)x_i.$$

Obviously, shown sum is equal to constant. Using equality (8.24) and taking into account that $\eta_{i1} = -1$, for this sum we will obtain the following expression,

$$H_{i1}^n - \binom{n}{2}H_{i1}^{n-1} + \left[2\binom{n}{3}+3\binom{n}{4}\right]H_{i1}^{n-2} + \ldots$$
$$\ldots + (-1)^{n-1}(n-1)!\,H_{i1}.$$

Condition (8.26) takes the form

$$\{H_{i1}^{n-1} - \binom{n}{2}H_{i1}^{n-2} + \left[2\binom{n}{3}+3\binom{n}{4}\right]H_{i1}^{n-3} + \ldots$$
$$\ldots + (-1)^{n-1}(n-1)!\}\,H_{i1} = 0. \qquad (8.28)$$

This equation has n - 1 roots differing from zero. We will designate them by symbols H_{ij} (i = 1, ..., n - 1).

In particular cases we have

a) during n = 2,

$$H_{11} = 1;$$

b) during n = 3,

$$H_{11} = 1,\ H_{21} = 2.$$

It remains for us to consider now a third case $\eta_{i1} < -1$. In this case in sum

$$x_i^n + d_1^{(1)}x_i^{n-1} + \ldots + d_{n-1}^{(1)}x_i$$

during $t \to \infty$ will prevail term

$$\eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2)x_i,$$

containing as a predominant component in the term

$$d_{n-1}^{(1)}x_i.$$

Since

$$\eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2) \neq 0, \qquad (8.29)$$

condition (8.26) can be executed only in the case when there is executed equality

$$\eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2)x_i + B_n t^{\rho_n+n} = 0. \qquad (8.30)$$

where magnitudes B_n and β_n for exponential coefficient b_n are determined by formulas (8.1), and in the case of compound-exponential coefficient b_n, given in the form (8.2) by equalities

$$B_n = \frac{B_{nl}}{C_{nm}}, \quad \beta_n = \beta_{nl} - \gamma_{nm}. \tag{8.31}$$

From equations (8.24) and (8.30) there follows

$$\eta_{i1} = \beta_n + n - 1, \quad H_{i1} = \frac{B_n}{\eta_{i1}(\eta_{i1}-1)\dots(\eta_{i1}-n+2)} = \\ = \frac{-B_n}{(\beta_n+n-1)(\beta_n+n-2)\dots(\beta_n+1)}. \tag{8.32}$$

Giving index i in this case value n and summing results with respect to determining of magnitudes $\varkappa_i$ in all considered cases, we will come to the following conclusion.

If there is executed condition (8.27), then equation (8.26) has n solutions. These solutions can be found as indicated above.

We will consider now the case when some of coefficients $b_i^{(1)}$ during $t \to \infty$ approach finite quantities, and for the remaining there is executed condition

$$\lim_{t\to\infty} b_i^{(1)} = 0. \tag{8.33}$$

In this case, as in the preceding, coefficients $d_j^{(1)}$ ($j = 1, \dots, n$) of polynomial $D_i^{(1)}(\varkappa_i)$ are bounded functions.

Having made, as earlier, three different assumptions about values of coefficient η_{i1}

$$\eta_{i1} > -1,$$
$$\eta_{i1} = -1,$$
$$\eta_{i1} < -1.$$

By analogy with the preceding case, we will come to the conclusion that the first possibility is excluded.

In the second case magnitude $\varkappa_i$ is constant and sum (8.23) during $t \to \infty$ approaches magnitude

$$x_i^n + \left[\binom{n}{2}\eta_{i1} + \lim_{t\to\infty} b_1^{(1)}\right]x_i^{n-1} + \dots + [\eta_{i1}(\eta_{i1}-1)\dots(\eta_{i1}- \\ -n+2) + \dots + \eta_{i1}\lim_{t\to\infty} b_{n-2}^{(1)} + \lim_{t\to\infty} b_{n-1}^{(1)}]x_i + \lim_{t\to\infty} b_n^{(1)}. \tag{8.34}$$

which by condition $\eta_{11} = -1$ is equal to magnitude

$$\varkappa_i^n+\left[\lim_{i\to\infty} b_1^{(1)}-\binom{n}{2}\right]\varkappa_i^{n-1}\cdot\ .\ .\ .\ +[\lim_{i\to\infty} b_{n-1}^{(1)}-\lim_{i\to\infty} b_{n-2}^{(1)}+$$
$$+2\lim_{i\to\infty} b_{n-3}^{(1)}+\ .\ .\ .\ +(-1)^{n-1}(n-1)!]\varkappa_i+\lim_{i\to\infty} b_n^{(1)}.$$

If

$$\lim_{i\to\infty} b_n^{(1)} \neq 0, \qquad (8.35)$$

then equating magnitude (8.34) to zero, we will obtain an equation containing n nonzero roots $\varkappa_i$ (among which can be multiples). Because of equation (8.24) these roots are equal to unknown values of magnitude H_{11}, i.e., magnitudes H_{11}, ..., H_{n1}.

If condition (8.35) is not executed, i.e., there occurs equality

$$\lim_{i\to\infty} b_n^{(1)} = 0. \qquad (8.36)$$

then two cases are possible:

$$\text{a) } \lim_{i\to\infty} b_{n-1}^{(1)}-\lim_{i\to\infty} b_{n-2}^{(1)}+\ .\ .\ .\ \mp(-1)^{n-1}(n-1)!\neq 0,$$

$$\text{b) } \lim_{i\to\infty} b_{n-1}^{(1)}-\lim_{i\to\infty} b_{n-2}^{(1)}\ .\ .\ .\ +(-1)^{n-1}(n-1)! = 0.$$

If limiting values of all coefficients $b_1^{(1)}$, ..., $b_{n-1}^{(1)}$, with the exception of any one, are fixed, and the value of one coefficient continuously changes in such a manner so that it passes along a number-scale axis from a sufficiently large negative number to a sufficiently large positive number, then case (b) will take place only during one value of this coefficient, and case (a) will occur during all the remaining values. Therefore, case (a) may be called model and case (b) special.

In the model, case due to condition (8.36), polynomial (8.34) has one zero root. Therefore, in this case there may be determined $n - 1$ values of magnitude H_{11}, i.e., H_{11}, ..., $H_{n-1,1}$.

If there occurs a special case where simultaneously with condition (8.36) there is executed still $k - 1$ equalities [where k is a natural number not smaller than unity and not larger than $(n - 1)$], at which coefficients of polynomial (8.34) during degrees $\varkappa_i$, $\varkappa_i^2$, ..., $\varkappa_i^{k-1}$ turn into zero, then number of nonzero roots of this polynomial decreases $n - k$. After determining these roots, we thereby will find $n - k$ values of magnitude H_{11}, i.e., H_{11}, ..., $H_{n-k,1}$.

During fulfillment of condition (8.35), the possibility of case $\eta_{11} = -1$ is excluded. Therefore, passing to a third case, we will assume that there occurs equality (8.36).

Let us consider binomial

$$d_{n-1}^{(i)}x_i+d_n^{(i)}.$$

Will replace coefficient $d_{n-1}^{(i)}$ by limiting value

$$\lim_{t\to\infty} b_{n-1}^{(i)}+\eta_{i1}\lim_{t\to\infty} b_{n-2}^{(i)}+\ldots+\eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2),$$

where for every $j = 1, \ldots, n - 1$ magnitude $\lim_{t\to\infty} b_j^{(i)}$ is equal either to zero or to coefficient b_j, corresponding to determination (8.1) in the case of exponential coefficient b_j, and equal to ratio

$$\frac{B_{jl}}{C_{jm}}$$

in the case of compound-exponential coefficient b_j [see formulas (8.2) and (8.4)]. Coefficient $d_n^{(i)}$ we will replace asymptotically by equivalent magnitude

$$B_n t^{\beta_n+n}.$$

After equating the magnitude to which the binomial is transformed to zero with these replacements, we will obtain equation

$$[\lim_{t\to\infty} b_{n-1}^{(i)}+\eta_{i1}\lim_{t\to\infty} b_{n-2}^{(i)}+\ldots+\eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2)]x_i+B_n t^{\beta_n+n}=0. \qquad (8.37)$$

Comparing equations (8.37) and (8.24), we will find

$$\eta_{i1}=\beta_n+n-1. \qquad (8.38)$$

Carrying out substitution (8.38) in expression for coefficients during x_i of equation (8.37), let us note that there are possible two cases:

a) $\lim_{t\to\infty} b_{n-1}^{(i)}+(\beta_n+n-1)\lim_{t\to\infty} b_{n-2}^{(i)}+\ldots+(\beta_n+n-1)(\beta_n+n-2)\ldots(\beta_n+1)\neq 0,$

b) $\lim_{t\to\infty} b_{n-1}^{(i)}+(\beta_n+n-1)\lim_{t\to\infty} b_{n-2}^{(i)}+\ldots+(\beta_n+n-1)(\beta_n+n-2)\ldots(\beta_n+1)=0.$

In first case, which it is possible to call model, having compared equations (8.37) and (8.24), we will obtain

$$H_{i1}=\frac{-B_n}{\lim_{t\to\infty} b_{n-1}^{(i)}+(\beta_n+n-1)\lim_{t\to\infty} b_{n-2}^{(i)}+\ldots+(\beta_n+n-1)(\beta_n+n-2)\ldots(\beta_n+1)}. \qquad (8.39)$$

Since during fulfillment of condition (a) in sum

$$x_i^n+d_1^{(i)}x_i^{n-1}+\ldots+d_{n-1}^{(i)}x_i \qquad (8.40)$$

during $t\to\infty$ there prevails the last term, then found values of coefficients η_{i1}

[formula (8.38)] and H_{11} [formula (8.39)] satisfy condition (8.26).

In the second case, which it is possible to consider in the class of special cases, the denominator of the right side of formula (8.39) turns into zero, as a consequence of which the presented method does not lead to determination of coefficient H_{11}.

Thus, during condition (8.36) the presented method gives the possibility of determining not more than one pair of values of coefficients H_{11} and η_{11}, for which $\eta_{11} < -1$. These values cannot be determined only in the special case when coefficients of an equation of oscillations are connected by the dependence shown in paragraph (b).

Summarizing this result with the result obtained above, we conclude that in the case when some coefficients $b_i^{(1)}$ during $t \to \infty$ approach finite quantities, and for others condition (8.33) is executed, equation (8.26) has not more than n solutions, where some of these solutions can be identical. If coefficients of an equation of oscillations do not satisfy conditions determining special cases, then equation (8.26) has exactly n solutions. Coefficients H_{11} and η_{11}, corresponding to solutions of equation (8.26) can be found by above-stated methods.

Let us turn to analysis of the last possible case at which some coefficients $b_i^{(1)}$ satisfy condition

$$\lim_{t\to\infty} b_i^{(1)} = \pm\infty, \tag{8.41}$$

i.e., are numerically increasing functions of time.

Having assumed

$$x_i = \nu_i t^{\xi} \tag{8.42}$$

and carrying out this substitution in expression (8.23), we will copy it in the form

$$D_i^{(1)}(x_i) = t^{n\xi}\left(\nu^n + \frac{d_1^{(1)}}{t^{\xi}}\nu^{n-1} + \dots + \frac{d_n^{(1)}}{t^{n\xi}}\right) = t^{n\xi}E^{(1)}(\nu). \tag{8.43}$$

Let us assume that ξ is minimum magnitude at which all coefficients of polynomial $E^{(1)}(\nu)$ during any real values of coefficient η_{11} are limited. Then in accordance with formulas for coefficients $d_j^{(1)}$ this magnitude should satisfy system of inequalities

$$\beta_j + j - j\xi \leqslant 0 \quad (j = 1, \dots, n). \tag{8.44}$$

where for one value of index j there must be executed equality

$$\beta_j + j - j\xi = 0. \tag{8.45}$$

Actually, formulas (8.20) express coefficient $d_j^{(1)}$ in the form of a linear sum of coefficients $b_1^{(1)}$, $b_2^{(1)}$, ..., $b_j^{(1)}$, where the coefficient during the last magnitude is equal to unity. The validity of the given conditions would be evident if in these sums there were absent all components besides the last ones. Therefore, it only must be shown that k-th component of j-th sum, where k and j are arbitrary, where k < j after division by $t^{j\xi}$, becomes a bounded function, i.e., there occurs inequality

$$\beta_k + k - j\xi \leqslant 0 \tag{8.46}$$

during k < j. But this can immediately be seen if one were to compare this inequality with k-th inequality of system (8.44)

$$\beta_k + k - k\xi \leqslant 0.$$

Further it is expedient to note that from the last inequality it follows that inequality (8.46) (during k < j) is an absolute inequality, i.e., has the form

$$\beta_k + k - j\xi < 0.$$

This means that after division by $t^{j\xi}$, all components of j-th sums, besides the last ones, are turned into vanishing functions. It follows from this that coefficients of polynomial $F^{(1)}(\nu)$ are either vanishing functions or nonvanishing, but limited, asymptotically equivalent to magnitudes

$$\frac{b_j^{(1)}}{t^{j\xi}}.$$

In accordance with inequalities (8.44) and equality (8.45) magnitude ξ may be determined by the formula

$$\xi = \max_j \frac{\beta_j + j}{j},$$

which it is possible to take for working, calculating formula.

We will look for exponential function

$$\nu = H'_{i1} t^{\eta'_{i1}+1} \tag{8.47}$$

(where H'_{i1} is constant complex coefficient, and η'_{i1} is constant real coefficient), which during substitution in polynomial (8.43) ensure fulfillment of condition (8.26). On the basis of dependence (8.42) they will determine simply exponential function $\pi_i(t)$, and on the basis of dependence (8.24) they will determine coefficients η_{i1} and H_{i1}.

Since factor $t^{n\xi}$ is not reflected on conditions (8.26), then coefficients H_{i1} and η_{i1} one can determine, analyzing polynomial $E_i^{(1)}(\nu)$ in exactly the same way as was analyzed polynomial $D_i^{(1)}(\varkappa)$. Considering

$$\frac{e_j^{(1)}}{t^{j\xi}} = e_j^{(1)} \qquad (j = 1, \ldots, n). \tag{8.48}$$

will reduce the problem to the second of the considered cases with the only difference that the formulas of connection of coefficients $e_j^{(1)}$ with coefficients η_{i1} and $b_j^{(1)}$ $(j = 1, \ldots, n)$ will be different.

We will establish these formulas.

We will determine preliminarily formulas of connection between coefficients H_{i1} and η_{i1} and coefficients H'_{i1} and η'_{i1}.

Having compared formulas (8.24), (8.42), (8.47), and (8.48), we will obtain

$$H_{i1} = H'_{i1}. \tag{8.49}$$

$$\eta_{i1} = \eta'_{i1} + \xi. \tag{8.50}$$

Using these formulas and formulas (8.20), (8.48), and (8.50), we will find

$$\left.\begin{aligned}
&e_1^{(1)} = \left[\binom{n}{2}(\eta'_{i1}+\xi) + b_1^{(1)}\right]t^{-\xi},\\
&e_2^{(1)} = \left[\left(\binom{n}{3}(\eta'_{i1}+\xi-1) + 3\binom{n}{4}(\eta'_{i1}+\xi)\right](\eta'_{i1}+\xi) + \\
&\quad + \binom{n-1}{2}(\eta'_{i1}+\xi)\, b_1^{(1)} + b_2^{(1)}\right]t^{-2\xi},\\
&e_3^{(1)} = \left\{\sum_{j=1}^{n-3} j\,(\eta'_{i1}+\xi)\,[(\eta'_{i1}+\xi-1)(\eta'_{i1}+\xi-2) + \right.\\
&\quad + 3(j-1)(\eta'_{i1}+\xi-1) + (j-1)(j-2)(\eta'_{i1}+\xi)^2] + \\
&\quad + \left[\binom{n-1}{3}(\eta'_{i1}+\xi-1) + 3\binom{n-1}{4}(\eta'_{i1}+\xi)\right](\eta'_{i1}+\xi)b_1^{(1)} + \\
&\quad \left. + \binom{n-2}{2}(\eta'_{i1}+\xi)\, b_2^{(1)} + b_3^{(1)}\right\}t^{-3\xi},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&e_{n-1}^{(1)} = [(\eta'_{i1}+\xi)(\eta'_{i1}+\xi-1)\ldots(\eta'_{i1}+\xi-n+2) + \ldots\\
&\quad \ldots + (\eta'_{i1}+\xi)b_{n-2}^{(1)} + b_{n-1}^{(1)}]\,t^{(1-n)\xi},\\
&e_n^{(1)} = b_n^{(1)}t^{-n\xi}.
\end{aligned}\right| \tag{8.51}$$

Let us consider polynomial

$$E^{(1)}(\nu) = \nu^n + e_1^{(1)}\nu^{n-1} + \ldots + e_n^{(1)}. \tag{8.52}$$

Due to condition of selection of magnitude ξ all its coefficients are limited, whatever the value of coefficient η'_{i1}. During $\eta'_{i1} > -1$ because of equality (8.47) it may be represented in the form

$$E^{(1)}(\nu) = (H'_{i1})^n t^{n(\eta'_{i1}+1)}[1 + o(1)].$$

In accordance with equality (8.43) condition (8.26) takes the form

$$H'_{i1} = 0.$$

But since $H'_{i1} = H_{i1}$, and the last magnitude, by definition, should be different from zero, the mentioned condition during shown values of coefficient η'_{i1} ($\eta'_{i1} > -1$) is not executed.

Let us consider the case $\eta'_{i1} = -1$.

Let us assume that n - k (where k may be an integer from zero to n - 1) is maximum value of index j, for which is executed condition

$$\xi = \frac{\beta_j + j}{j}. \tag{8.53}$$

Then there occurs equality

$$\lim_{t\to\infty} e^{(i)}_{n-k} = \lim_{t\to\infty} \frac{b^{(i)}_{n-k}}{t^{(n-k)\xi}} = \text{const} \neq 0.$$

and condition (8.26) is executed for n - k values of coefficient H'_{i1} which are roots of equation

$$(H'_{i1})^{n-k} + (H'_{i1})^{n-k-1} \lim_{t\to\infty} e^{(i)}_1 + \ldots + \lim_{t\to\infty} e^{(i)}_{n-k} = 0. \tag{8.54}$$

Coefficients of this equation $e^{(i)}_j$ ($j = 1, \ldots, n - k$) have the form

$$e^{(i)}_j = \lim_{t\to\infty} \frac{b^{(i)}_j}{t^{j\xi}}.$$

Solving equation (8.54), we will find n - k unknown values of coefficient H'_{i1}. These values, jointly with initial condition $\eta'_{i1} = -1$, because of formulas (8.49) and (8.50), will determine n - k pairs of values of coefficients η_{i1} and H_{i1}.

Besides found values of coefficients η'_{i1} and H'_{i1}, conditions (8.26) in model cases are satisfied still by k pairs of values of coefficients η'_{i1} and H'_{i1}, where for all these pairs $\eta'_{i1} < -1$. These values can be fixed from consideration of polynomial

$$\nu^k \lim_{t\to\infty} e^{(i)}_{n-k} + e^{(i)}_{n-k+1} \nu^{k-1} + \ldots + e^{(i)}_{n-1} \nu + B_n t^{\beta'_n + \eta}. \tag{8.55}$$

where

$$\beta'_n = \beta_n + n\xi. \tag{8.56}$$

Carrying out substitution

$$\nu = \mu t^{-\xi}, \tag{8.57}$$

will give polynomial (8.55) the form

$$t^{-k\xi_1}\Big(\rho^k \lim_{t\to\infty} e^{(i)}_{n-k} + e^{(i)}_{n-k+1} t^{\xi_1}\rho^{k-1} + \ldots + e^{(i)}_{n-1} t^{(k-1)\xi_1}\rho +$$

$$+ B_n t^{\beta'_n + n + k\xi_1}\Big) = t^{-k\xi_1} F^{(1)}(\rho). \tag{8.58}$$

We will select coefficient ξ_1 so that all coefficients of polynomial $F^{(1)}(\rho)$ during any values of coefficient η'_{i1} are bounded functions, but so that at least one of them, not counting coefficient during degree ρ^k, is constant. In other words, after determining coefficient β'_j as minimum real magnitude for which during any values of coefficient η_{i1} there occurs equality

$$e_j^{(i)} = O\left(t^{\beta'_j + j}\right). \tag{8.59}$$

selection of coefficient ξ_1 we will subordinate to system of inequalities

$$\beta'_{n-k+j} + n - k + j(1 + \xi_1) \leqslant 0 \quad (j = 1, \ldots, k) \tag{8.60}$$

and equality

$$\beta'_{n-k+j} + n - k + j(1 + \xi_1) = 0, \tag{8.61}$$

fulfillment of which is demanded at least for one value of j. With these conditions coefficient ξ_1 is determined uniquely. Formula for its determination has the form

$$\xi_1 = \min_j\left(-\frac{\beta'_{n-k+j} + n - k + j}{j}\right). \tag{8.62}$$

Let us assume that k_1 is maximum from indices $j = 1, 2, \ldots, k$ for which is executed equality (8.61). Then we will constitute equation

$$\rho^{k_1} \lim_{t\to\infty} e^{(i)}_{n-k} + \rho^{k_1-1} \lim_{t\to\infty} e^{(i)}_{n-k+1} t^{\xi_1} + \ldots$$

$$\ldots + \rho \lim_{t\to\infty} e^{(i)}_{n-k+k_1} t^{\beta'_{n-k+k_1} + n - k + k_1 + k_1\xi} = 0, \tag{8.63}$$

after determining value of coefficient η'_{i1} from condition ρ = const, i.e., equality

$$\eta'_{i1} = -1 - \xi_1 \tag{8.64}$$

[compare formulas (8.47) and (8.57)]. In the <u>model case</u>, when

$$\lim_{t\to\infty} e^{(i)}_{n-k+k_1} t^{\beta'_{n-k+k_1} + n - k + k_1 + k_1\xi} \neq 0, \tag{8.65}$$

all roots of this equation are nonzero. In the <u>special case</u> when shown inequality is turned into equality, equation (8.63) obtains zero roots, in conformity with the number of which decreases the number of its nonzero roots.

After determining values of coefficient H'_{ij} as nonzero roots of equation (8.63) and taking into account that during condition

$$\eta'_{i1} < -1$$

in polynomial

$$v^{n} + e_1^{(i)} v^{n-1} + \dots + e_k^{(i)} v^{n-k} \tag{8.66}$$

during $t \to \infty$ the last term prevails, we will conclude that fixed values of coefficients H'_{i1} in combination with value of coefficient η'_{i1}, determined by formula (8.64), determine k_1 or less pairs of values of these coefficients, satisfying condition (8.26). By formulas (8.49) and (8.50) can be found (corresponding to them) values of coefficients H_{i1} and η_{i1}

$$H_{i1} = H'_{i1},$$
$$\eta_{i1} = \xi - \xi_1 - 1.$$

If $k_1 = k$, then in equation (8.63) coefficient

$$\lim_{t \to \infty} e^{(i)}_{n-k+k_1} t^{\beta'_{n-k+k_1}+n-k+k_1+k_1\xi}$$

one should replace by coefficient B_n. In the model case found values of coefficients η_{i1} and H_{i1} will supplement system of their earlier found values in such a way that the total number of found pairs of values of these coefficients will be equal to n. With this, all possibilities of determining these coefficients will be exhausted.

If $k_1 < k$, then for further determination of coefficients η_{i1} and H_{i1}, one should consider polynomial

$$\rho^{k-k_1} \lim_{t \to \infty} e^{(i)}_{n-k+k_1} t^{\beta'_{n-k+k_1}+n-k+k_1+k_1\xi} + e^{(i)}_{n-k+k_1+1} t^{\beta'_{n-k+k_1+1}+n-k+k_1+1+(k_1+1)\xi_1} \rho^{k-k_1-1} + \dots + B_n t^{\beta'_n+n+k\xi_1}. \tag{8.67}$$

Introducing substitution

$$\rho = \sigma t^{-\xi_1}, \tag{8.68}$$

where

$$\xi_2 = \min_j \left[-\frac{n-k+k_1+j+(k_1+j)\xi_1}{j} \right] \quad (j = 1, 2, \dots, k-k_1), \tag{8.69}$$

will give polynomial (8.67) the form

$$t^{(k_1-k)\xi_1} \Big(\sigma^{k-k_1} \lim_{t \to \infty} e^{(i)}_{n-k+k_1} t^{\beta'_{n-k+k_1}+n-k+k_1+k_1\xi} + e^{(i)}_{n-k+k_1+1} t^{\beta'_{n-k+k_1+1}+n-k+k_1+1+(k_1+1)\xi_1-\xi_1} \sigma^{k-k_1-1} + \dots + B_n t^{\beta'_n+n+k\xi_1} \Big).$$

in which all components in parentheses are products of certain degree σ and of coefficients which are bounded functions; it is always possible to indicate such values of coefficient η'_{i1} at which first component is a constant different from zero and at least one of the other components approaches a limit different from zero.

Replacing coefficients of polynomial included in parentheses by their limiting values and considering

$$\eta_{i1} = \xi - \xi_1 - \xi_2 - 1, \tag{8.70}$$

which stipulates constancy of magnitude σ, equating this polynomial to zero, we will obtain algebraic equation about magnitude σ.

Let us assume that k_2 is maximum index for which is executed equality

$$\xi_2 = -\frac{n-k+k_1+j+(k_1+j)\xi_1}{j},$$

and let us assume that coefficients of mentioned polynomial during degrees σ^{k-k_1} and $\sigma^{k-k_1-k_2}$, during condition (8.70) are different than zero. Then, solving obtained equation, one can determine k_2 of values σ which are constant and different from zero and which, in turn, because of equalities (8.68), (8.57), (8.47), and (8.26) will determine k_2 of values of coefficients H_{i1}. These values of coefficients H_{i1} in combination with value (8.70) of coefficient η_{i1} will give k_2 of pairs of values of these coefficients, satisfying condition (8.26).

Cases when coefficients, one or both, during powers σ^{k-k_1} and $\sigma^{k-k_1-k_2}$ turn into zero, one should relate to the class of special cases. In this case the number of additional pairs of values of coefficients η_{i1} and H_{i1} found by the presented method, less than k_2 and in particular cases may be equal to zero.

If $k_2 = k - k_1$, then coefficient during power $\sigma^{k-k_1-k_2}$ is equal to coefficient b_n and, consequently, is a constant differing from zero. Under this condition, a special case can take place only when coefficient during power σ^{k-k_1} becomes zero. If this does not occur and also during determination of earlier found values of coefficients η_{i1} and H_{i1} special cases were not encountered, then k_2 pairs of values of coefficients η_{i1} and H_{i1} found by the shown method will supplement the system of earlier found values up to a system consisting of n pairs. With this, all possibilities of determining these coefficients will be exhausted.

If $k_2 < k - k_1$, then further determination of coefficients η_{i1} and H_{i1} can be conducted in an analogous way. Since the number of terms of an equation of free oscillations is finite, then there will occur a moment when process of determining

coefficients will be completed. With this, the number of found pairs of their values will be equal to or less than n.

Finishing this account of a method of determining coefficients η_{i1} and H_{i1}, we will conclude that, independently of form of coefficients of equations of free oscillations of the class considered in this chapter, the number of different pairs of their values and also the number of pairs of their values, calculated taking into account multiplicity of roots, will not exceed n, where the latter number will be exactly equal to n in all cases which do not lead to special cases. These values can be determined by the above-stated method.

§ 4. Method of Determining Coefficients H_{12}, H_{13}, ... and η_{12}, η_{13}, ...

In the preceding paragraph was presented a method of finding coefficients H_{i1} and η_{i1} ($i = 1, \ldots, n$), determining the first component of sums

$$\sum_{j=1}^{\infty} H_{ij} t^{\eta_{ij}},$$

appearing in formula (8.8) of suggested general solution of equation of free oscillations with exponential or with compound-exponential coefficients. In this paragraph is considered a method of determining analogous coefficients, determining subsequent component, on the assumption that coefficients H_{i1} and η_{i1} already are found.

We will compare i-th component of sum (8.8)

$$x_i(t) = C_i \exp \int^{t} \sum_{j=1}^{\infty} H_{ij} t^{\eta_{ij}} dt \tag{8.71}$$

with approximate solution

$$\tilde{x}_i(t) = C_i \exp \int^{t} \zeta_i^{(1)}(t)\, dt, \tag{8.72}$$

after determining function $\zeta_i^{(1)}(t)$ by equality

$$\zeta_i^{(1)}(t) = H_{i1} t^{\eta_{i1}} + H_{i2} t^{\eta_{i2}}, \tag{8.73}$$

in which H_{i1} and η_{i1} – known coefficients, satisfying condition (8.26) and H_{i2} and η_{i2} – searched coefficients. Possible values of coefficients η_{i2} we will limit by condition

$$\eta_{i2} < \eta_{i1}. \tag{8.74}$$

We will assume that functions $x_i(t)$ and $\tilde{x}_i(t)$ are equal to each other, where the first of them is solution of equation (8.1). Then function $\zeta_i^{(1)}(t)$ during all t should satisfy equality

$$(\zeta_i^{(1)}+D)^{n-1}\zeta_i^{(1)}+[(\zeta_i^{(1)}+D)^{n-2}\zeta_i^{(1)}]b_1+\ldots+b_n=0. \qquad (8.75)$$

from which, carrying out substitution (8.73), one can determine coefficients H_{12} and η_{12}.

If functions $x_i(t)$ and $\tilde{x}_i(t)$ are not connected by sign of equality, but the first of them is solution of equation (0.1), then left part of equation (8.75) is different than zero and is a certain measure of approximation of approximate solution $\tilde{x}_1(t)$ to exact. In this case one can determine such values of coefficients H_{12} and η_{12} at which, in a certain meaning, there is attained maximum accuracy of approximation of left part of equality (8.75) to zero.

For determination of unknown values of coefficients H_{12} and η_{12} in both mentioned cases there may be recommended a single method. This method and also its theoretical foundation are presented below.

Differentiating equality (8.73) one, two, ..., n - 1 times, we will obtain relationship

$$\left.\begin{aligned}
\frac{d\zeta_i^{(1)}}{dt} &= \eta_{i1}\zeta_i^{(0)}t^{-1}+\eta_{i2}\Delta\zeta_i^{(0)}t^{-1},\\
\frac{d^2\zeta_i^{(1)}}{dt^2} &= \eta_{i1}(\eta_{i1}-1)\zeta_i^{(0)}t^{-2}+\eta_{i2}(\eta_{i2}-1)\Delta\zeta_i^{(0)}t^{-2},\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
\frac{d^{n-1}\zeta_i^{(1)}}{dt^{n-1}} &= \eta_{i1}(\eta_{i1}-1)\ldots(\eta_{i1}-n+2)\zeta_i^{(0)}t^{1-n}+\\
&\quad+\eta_{i2}(\eta_{i2}-1)\ldots(\eta_{i2}-n+2)\Delta\zeta_i^{(0)}t^{1-n},
\end{aligned}\right\} \qquad (8.76)$$

where

$$\Delta\zeta_i^{(0)}=\zeta_i^{(1)}-\zeta_i^{(0)}=H_{i2}t^{\eta_{i2}}. \qquad (8.77)$$

Because of relationships (8.76) and (8.77) it is possible to record left part of equation (8.75) in the form

$$\begin{aligned}
&(\zeta_i^{(0)}+D)^{n-1}\zeta_i^{(0)}+b_1(\zeta_i^{(0)}+D)^{n-2}\zeta_i^{(0)}+\ldots+b_n+\\
&+(\zeta_i^{(0)}+D)^{n-1}\Delta\zeta_i^{(0)}+b_1(\zeta_i^{(0)}+D)^{n-2}\Delta\zeta_i^{(0)}+\ldots+b_{n-1}\Delta\zeta_i^{(0)}+\\
&+\Delta\zeta_i^{(0)}(\zeta_i^{(0)}+D)^{n-2}\zeta_i^{(0)}+b_1\Delta\zeta_i^{(0)}(\zeta_i^{(0)}+D)^{n-3}\zeta_i^{(0)}+\ldots\\
&\ldots+b_{n-2}\Delta\zeta_i^{(0)}\zeta_i^{(0)}+(\zeta_i^{(0)}+D)\Delta\zeta_i^{(0)}(\zeta_i^{(0)}+D)^{n-3}\zeta_i^{(0)}+\ldots\\
&\ldots+b_{n-3}(\zeta_i^{(0)}+D)\Delta\zeta_i^{(0)}\zeta_i^{(0)}+\ldots+(\zeta_i^{(0)}+D)^{n-2}\Delta\zeta_i^{(0)}\zeta_i^{(0)}+R^{(1)}.
\end{aligned} \qquad (8.78)$$

Here $R^{(1)}$ is sum of terms containing second and highest powers of $\zeta_i^{(0)}$. Sum

$$S_1^{(1)}=(\zeta_i^{(0)}+D)^{n-1}\Delta\zeta_i^{(0)}+b_1(\zeta_i^{(0)}+D)^{n-2}\Delta\zeta_i^{(0)}+\ldots+b_{n-1}\Delta\zeta_i^{(0)},$$
$$S_2^{(1)}=\Delta\zeta_i^{(0)}(\zeta_i^{(0)}+D)^{n-2}\zeta_i^{(0)}+b_1(\Delta\zeta_i^{(0)}+D)^{n-3}\Delta\zeta_i^{(0)}+\ldots+b_{n-2}\Delta\zeta_i^{(0)}\zeta_i^{(0)},$$
$$S_3^{(1)}=(\zeta_i^{(0)}+D)\Delta\zeta_i^{(0)}(\zeta_i^{(0)}+D)^{n-3}\zeta_i^{(0)}+\ldots+b_{n-3}(\zeta_i^{(0)}+D)\Delta\zeta_i^{(0)}\zeta_i^{(0)},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_n^{(1)}=(\zeta_i^{(0)}+D)^{n-2}\Delta\zeta_i^{(0)}\zeta_i^{(0)},$$

entering as components into expression (8.78), it is possible to give the form

$$\left.\begin{aligned}
S_1^{(1)}&=(\zeta_i^{(0)})^{n-1}\Delta\zeta_i^{(0)}+\left[\binom{n-1}{2}\tau_{i1}+(n-1)\tau_{i2}+\right.\\
&\left.+b_1^{(1)}\right]t^{-1}(\zeta_i^{(0)})^{n-2}\Delta\zeta_i^{(0)}+\left\{\sum_{\substack{j,k=0\\ j+k<n-3}}^{n-3}(j\tau_{i1}+\tau_{i2})[(n-\right.\\
&-3-k)\tau_{i1}+\tau_{i2}-1]+b_1^{(1)}\left[\binom{n-1}{2}\tau_{i1}+\tau_{i2}-\tau_{i1}\right]+\\
&\left.+b_2^{(1)}\right\}t^{-2}(\zeta_i^{(0)})^{n-3}\Delta\zeta_i^{(0)}+\ldots+[\tau_{i2}(\tau_{i2}-1)\ldots(\tau_{i2}-n+2)+\\
&+b_1^{(1)}\tau_{i2}(\tau_{i2}-1)\ldots(\tau_{i2}-n+3)+\ldots+b_{n-1}^{(1)}]t^{1-n}\Delta\zeta_i^{(0)},\\
S_2^{(1)}&=(\zeta_i^{(0)})^{n-1}\Delta\zeta_i^{(0)}+\left[\binom{n-1}{2}\tau_{i1}+b_1^{(1)}\right]t^{-1}(\zeta_i^{(0)})^{n-2}\Delta\zeta_i^{(0)}+\\
&+\left\{\tau_{i1}\sum_{\substack{j,k=0;\\ j+k<n-4}}^{n-4}(j+1)[(n-3-k)\tau_{i1}-1]+\right.\\
&\left.+b_1^{(1)}\binom{n-2}{2}\tau_{i1}+b_2^{(1)}\right\}t^{-2}(\zeta_i^{(0)})^{n-3}\Delta\zeta_i^{(0)}+\ldots\\
&\ldots+[\tau_{i1}(\tau_{i1}-1)\ldots(\tau_{i1}-n+3)+b_1^{(1)}\tau_{i1}(\tau_{i1}-1)\ldots\\
&\ldots(\tau_{i1}-n+4)+\ldots+b_{n-2}^{(1)}]\zeta_i^{(0)}t^{2-n}\Delta\zeta_i^{(0)},\\
S_3^{(1)}&=(\zeta_i^{(0)})^{n-1}\Delta\zeta_i^{(0)}+[\tau_{i2}+\tau_{i1}(n-2)+\tau_{i1}\binom{n-2}{2}+\\
&+b_1^{(1)}]t^{-1}(\zeta_i^{(0)})^{n-2}\Delta\zeta_i^{(0)}+\left\{\tau_{i1}[\tau_{i2}+\tau_{i1}(n-3)-1]\binom{n-2}{2}+\right.\\
&+\tau_{i1}\sum_{\substack{j,k=0;\\ j+k<n-5}}^{n-5}(j+1)[(n-4-k)\tau_{i1}-1]+\\
&+b_1^{(1)}\left[\tau_{i2}+\tau_{i1}(n-3)+\tau_{i1}\binom{n-3}{2}\right]+\\
&\left.+b_2^{(1)}\right\}t^{-2}(\zeta_i^{(0)})^{n-3}\Delta\zeta_i^{(0)}+\ldots+[\tau_{i1}(\tau_{i1}-1)\ldots\\
&\ldots(\tau_{i1}-n+4)(\tau_{i1}+\tau_{i2}-3+n)+b_1^{(1)}\tau_{i1}(\tau_{i1}-1)\ldots\\
&\ldots(\tau_{i1}-n+5)(\tau_{i1}+\tau_{i2}-4+n)+\ldots\\
&\ldots+b_{n-3}^{(1)}(\tau_{i1}+\tau_{i2})]t^{2-n}\zeta_i^{(0)}\Delta\zeta_i^{(0)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
S_n^{(1)}&=\left\{(\zeta_i^{(0)})^{n-1}+\left[\binom{n-1}{2}\tau_{i1}+(n-2)\tau_{i2}\right]t^{-1}(\zeta_i^{(0)})^{n-2}+\right.\\
&+\sum_{\substack{j,k=0;\\ j+k<n-4}}^{n-4}[(j+1)\tau_{i1}+\tau_{i2}][(n-k-3)\tau_{i1}+\\
&+\tau_{i2}-1]t^{-2}(\zeta_i^{(0)})^{n-3}+\ldots+(\tau_{i1}+\tau_{i2})\ldots\\
&\left.\ldots(\tau_{i1}+\tau_{i2}-n+3)t^{2-n}\zeta_i^{(0)}\right\}\Delta\zeta_i^{(0)}/
\end{aligned}\right\}\qquad(8.79)$$

We will designate

$$(\zeta_i^{(0)}+D)^{n-1}\zeta_i^{(0)}+b_1(\zeta_i^{(0)}+D)^{n-2}\zeta_i^{(0)}+\ldots+b_n=S_0^{(1)},$$
$$\frac{S_1^{(1)}+S_2^{(1)}+\ldots+S_n^{(1)}}{\Delta\zeta_i^{(0)}}=S_\Delta^{(1)}. \tag{8.80}$$

According to formula (8.79) magnitudes $S_\Delta^{(1)}$ is exponential or compound-exponential function of t and is expressed in a form not containing magnitude $\Delta\zeta_i^{(0)}$:

$$S_\Delta^{(1)}=n(\zeta_i^{(0)})^{n-1}+\left\{\binom{n}{2}[(n-2)\tau_{i1}+\tau_{i2}]+\right.$$
$$\left.+(n-1)b_1^{(1)}\right\}t^{-1}(\zeta_i^{(0)})^{n-2}+\left\{\left[\frac{3}{2}\binom{n}{4}-\frac{1}{2}\binom{n-2}{2}+\sum_{\substack{j+k=0\\ j+k<n-3}}^{n-3}j(n-3-k)\right.\right.$$
$$\left.+\sum_{m=0}^{n-4}\sum_{j+k=0}^{m}(j+1)(2m-2k+2)\right]\tau_{i1}^2+\ldots+(n-2)b_2^{(1)}\Big\}\times$$
$$\times t^{-2}(\zeta_i^{(0)})^{n-3}+\ldots+\left\{(n-2)!\left[\binom{\tau_{i1}}{n-2}+2\binom{\tau_{i1}+\tau_{i2}}{n-2}+\binom{\tau_{i2}}{n-2}\right]+\right.$$
$$\left.+\sum_{j=1}^{n-3}j!\,(n-2-j)!\left[\binom{\tau_{i1}}{j}+\binom{\tau_{i2}}{j}\right]\binom{\tau_{i1}+\tau_{i2}-j}{n-2-j}+\ldots+2b_{n-2}^{(1)}\right\}t^{2-n}\zeta_i^{(0)}+$$
$$+\left[\binom{\tau_{i2}}{n-1}(n-1)!+\binom{\tau_{i2}}{n-2}(n-2)!\,b_1^{(1)}+\ldots+\tau_{i2}b_{n-2}^{(1)}+b_{n-1}^{(1)}\right]t^{1-n}. \tag{8.81}$$

In particular, during n = 2, formula (8.81) has the form

$$S_\Delta^{(1)}=2\zeta_i^{(0)}+(\tau_{i2}+b_1^{(1)})t^{-1},$$

during n = 3, the form

$$S_\Delta^{(1)}=3(\zeta_i^{(0)})^2+[3(\tau_{i1}+\tau_{i2})+2b_1^{(1)}]t^{-1}\zeta_i^{(0)}+$$
$$+[\tau_{i2}(\tau_{i2}-1+b_1^{(1)})+b_2^{(1)}]t^{-2}.$$

Using introduced designations, it is possible to give expression (8.78) the form

$$S_0^{(1)}+S_\Delta^{(1)}\Delta\zeta_i^{(0)}+R^{(1)}. \tag{8.82}$$

In this expression magnitudes $S_0^{(1)}$, $S_\Delta^{(1)}$ and $R^{(1)}$ are exponential or compound-exponential functions of t (excluding degenerated cases, in which all or some of them turn into zeroes). According to the property of these functions, they can be represented in the form

$$\left.\begin{aligned} S_0^{(1)}&=M_0^{(1)}[t^{\mu}+o(t^{\mu})],\\ S_\Delta^{(1)}&=M_\Delta^{(1)}\left[t^{\mu_\Delta^{(1)}}+o\left(t^{\mu_\Delta^{(1)}}\right)\right],\\ R^{(1)}&=M_R^{(1)}\left[t^{\mu_R^{(1)}}+o\left(t^{\mu_R^{(1)}}\right)\right]. \end{aligned}\right\} \tag{8.83}$$

In expression for function $S_0^{(1)}(t)$ coefficients $M_0^{(1)}$ and μ_0 are known magnitudes. In expression for function $S_\Delta^{(1)}(t)$ – coefficients $M_\Delta^{(1)}$ and $\mu_\Delta^{(1)}$ depend on coefficient η_{i2}. Coefficients $M_R^{(1)}$ and $\mu_R^{(1)}$ in expression for function $R^{(1)}(t)$ depend on coefficients η_{i2} and H_{i2}, since function $R^{(1)}(t)$ depends on magnitude $\zeta_i^{(0)}(t)$.

We will designate

$$\left.\begin{aligned} m(\mu_\Delta^{(1)}) &= \max_{H_{i1},\eta_{i2}} \mu_\Delta^{(1)} \\ m(\mu_R^{(1)}) &= \max_{\eta_{i2},H_{i2}} \mu_R^{(1)}. \end{aligned}\right\} \qquad (8.84)$$

Disregarding in expression (8.82) magnitude $R^{(1)}$ and considering only highest powers of t, we will define magnitude η_{i2} from condition

$$M_0^{(1)} t^{\mu_0} + M_\Delta^{(1)} t^{m(\mu_\Delta^{(1)})} \Delta\zeta_i^{(0)} = 0. \qquad (8.85)$$

From equality (8.85) follows

$$\eta_{i2} = \mu_0 - m(\mu_\Delta^{(1)}). \qquad (8.86)$$

Two cases are possible:

a) model

$$m(\mu_\Delta^{(1)}) = \mu_\Delta^{(1)},\ M_\Delta^{(1)} \neq 0$$

b) special

$$m(\mu_\Delta^{(1)}) = \mu_\Delta^{(1)} \text{ when } M_\Delta^{(1)} = 0$$

Considering model case, for value η_{i2}, from equality (8.85) we will obtain

$$H_{i2} = -\frac{M_0^{(1)}}{M_\Delta^{(1)}}. \qquad (8.87)$$

Will show that in the model case during selection of coefficients η_{i2} and H_{i2} by formulas (8.86) and (8.87)

a) there is executed condition (8.74),

b) magnitudes $S_0^{(1)}$ and $-S_\Delta^{(1)}\Delta\zeta_i^{(0)}$ are asymptotically equivalent.

Actually, maximum exponent t in components forming sums $S_j^{(1)}$ ($j = 1, \ldots, n$) is no less than maximum exponent in components of each of the shown sums. Considering sum $S_1^{(1)}$, its corresponding maximum exponent we will designate by symbol μ'. According to that said above, we will write

$$m(\mu_\Delta^{(1)}) + \eta_{i2} \geq \mu'. \qquad (8.88)$$

On the other hand, according to conditions of selection of coefficients η_{11} and H_{11}, coefficient during maximum power of t in sum $S_0^{(1)}$ turns into zero, where coefficient β_n does not exceed maximum exponent t in component of sum $S_0^{(1)}$ -1_n, which, obviously, is equal to magnitude $\mu' - \eta_{12} + \eta_{11}$. Therefore

$$\mu_0 < \mu' - \eta_{12} + \eta_{11}. \tag{8.89}$$

Comparing inequalities (8.88) and (8.89), we will find

$$\mu_0 < m(\mu_\Delta^{(1)}) + \eta_{11}.$$

Hence because of equality (8.86) there follows inequality (8.74).

Passing to proof of second affirmation, let us note that left parts of equalities (8.75) and (8.15) it is possible to present in the form of sums

$$(\zeta_i^{(1)})^n + \binom{n}{1}(\zeta_i^{(1)})^{n-1} D\zeta_i^{(1)} + \ldots + D^{n-1}\zeta_i^{(1)} + \ldots + b_n$$

and

$$(\zeta_i^{(0)})^n + \binom{n}{1}(\zeta_i^{(0)})^{n-1} D\zeta_i^{(0)} + \ldots + D^{n-1}\zeta_i^{(0)} + \ldots + b_n.$$

The difference of each pair of corresponding components of these sums (with the exception of the difference of the last pair, which is equal to zero), after realization of operations of differentiation and transformations by formulas (8.16) and (8.76), may be located in a finite series in whole, positive, increasing powers of magnitude $\Delta\zeta_i^{(0)}$. Because of condition (8.74) exponents of exponential functions, asymptotically equivalent during $t \to \infty$ to the second and following component of each such sum, will be less than the exponents of exponential functions asymptotically equivalent to their first component, consequently, less than magnitude

$$m(\mu_\Delta^{(1)}) + \eta_{12}.$$

Therefore, during $t \to \infty$

$$R^{(1)} = o(S_\Delta^{(1)} \Delta\zeta_i^{(0)}) = o(S_0^{(1)}).$$

The validity of the proven affirmation follows from this.

From asymptotic equivalence of magnitudes $S_0^{(1)}$ and $-S_\Delta^{(1)}\Delta\zeta_i^{(0)}$ it follows that during determination of magnitudes η_{12} and H_{12} by formulas (8.86) and (8.87), in the model case the left part of equality (8.78) is asymptotically equivalent to certain magnitude

$$M_0^{(2)} t^{\mu_1},$$

in which exponent μ_1 is connected with exponent μ_0 by inequality

$$\mu_1 < \mu_0. \tag{8.90}$$

Because of this inequality during $t \to \infty$ approximate solution (8.72) more closely approaches exact than approximate solution

$$\tilde{x}_i(t)=C_i \exp\int_{T}^{t} \zeta_i^{(0)}(t)\,dt. \tag{8.91}$$

This property is the basis of the expediency of the shown method of selecting coefficients η_{12} and H_{12} (in model case).

During determination of coefficients η_{12} and H_{12} by formulas (8.86) and (8.87) there can take place one of two cases:

a) equality (8.75) is executed,

b) equality (8.75) is not executed.

If first case occurs, then function $\tilde{x}_1(t)$ (8.72) is solution of equation (0.1). If second case occurs, then one should consider the possibility of further more precise definition of approximate solution $\tilde{x}_1(t)$.

In light of the above-stated, for further more precise definition of approximate solution $\tilde{x}_1(t)$, it is necessary to determine in the appropriate way coefficients H_{ij} and η_{ij} during $j = 3, 4, \ldots$. There is set forth below, with the necessary foundations, a method of solving this problem and conditions are clarified with which this method leads to correct results.

Being based on the principle of full mathematical induction and being limited by consideration of model cases, we will consider the problem of determining coefficients H_{ij} and η_{ij} during $j = 3, 4, \ldots$ solved when there is shown a method not depending on value of index k, allowing us to find coefficients $H_{i,k+1}$ and $\eta_{i,k+1}$ or to establish the presence of a special case, if for $j \leq k$ are determined all coefficients H_{ij} and η_{ij}. It is clear, this problem we will consider solved also in the case when necessity for determination of coefficients $H_{i,k+1}$ and $\eta_{i,k+1}$ during certain k drops in connection with the fact that approximate solution

$$\exp\int_{T}^{t}\sum_{j=1}^{k} H_{ij}t^{\eta_{ij}}\,dt \tag{8.92}$$

coincides with exact.

We will discuss the method of determining coefficients H_{ij} and η_{ij} ($j = 3, 4, \ldots$) by recurrent dependences

$$\left.\begin{aligned}\eta_{i,k+1}&=\mu_{k-1}-m(\mu_k^{(k)}),\\ H_{i,k+1}&=-\frac{M_0^{(k)}}{M_k^{(k)}},\end{aligned}\right\} \tag{8.93}$$

the right sides of which are determined, as shown below, by coefficients $b_1, \ldots, b_n$:

$H_{i1}, \ldots, H_{ik}$ and $\eta_{i1}, \ldots, \eta_{ik}$.

We will explain the meaning and origin of these dependences.

Let us assume that coefficients H_{ij} and η_{ij} are found during $j \leq k$ with which are carried out conditions

$$H_{ij} \neq 0,$$
$$\eta_{ij} < \eta_{i,j-1}$$
$$(j = 1, 2, \ldots, k),$$

and function (8.92) is not an exact solution of equation (0.1). Then we will designate

$$\zeta_i^{(k-1)}(t) = \sum_{j=1}^{k} H_{ij} t^{\eta_{ij}} \tag{8.94}$$

and

$$\Delta\zeta_i^{(k-1)}(t) = H_{i,k+1} t^{\eta_{i,k+1}} \tag{8.95}$$

and, after limiting possible values of coefficient $\eta_{i,k+1}$ by inequality

$$\eta_{i,k+1} < \eta_{ik}, \tag{8.96}$$

we will present magnitude

$$S_0^{(k+1)} = (\zeta_i^{(k)} + D)^{n-1}\zeta_i^{(k)} + [(\zeta_i^{(k)} + D)^{n-2}\zeta_i^{(k)}] b_1 + \ldots + b_n,$$

being an indirect appraisal of the accuracy of approximation of function

$$\exp \int_{\tau}^{t} \zeta_i^{(k)}(t)\, dt$$

to solution, in the form of sum

$$S_0^{(k)} + S_1^{(k)} + \ldots + S_n^{(k)} + R^{(k)},$$

where

$$S_0^{(k)} = (\zeta_i^{(k-1)} + D)^{n-1}\zeta_i^{(k-1)} + [(\zeta_i^{(k-1)} + D)^{n-2}\zeta_i^{(k-1)}] b_1 + \ldots + b_n;$$
$$S_1^{(k)} = (\zeta_i^{(k-1)} + D)^{n-1}\Delta\zeta_i^{(k-1)} + b_1(\zeta_i^{(k-1)} + D)^{n-2}\Delta\zeta_i^{(k-1)} + \ldots + b_{n-1}\Delta\zeta_i^{(k-1)};$$
$$S_2^{(k)} = \Delta\zeta_i^{(k-1)}(\zeta_i^{(k-1)} + D)^{n-2}\zeta_i^{(k-1)} + b_1(\Delta\zeta_i^{(k-1)} + D)^{n-3}\Delta\zeta_i^{(k-1)} + \ldots$$
$$\ldots + b_{n-2}\Delta\zeta_i^{(k-1)}\zeta_i^{(k-1)};$$
$$S_3^{(k)} = (\zeta_i^{(k-1)} + D)\Delta\zeta_i^{(k-1)}(\zeta_i^{(k-1)} + D)^{n-3}\zeta_i^{(k-1)} + \ldots + b_{n-3}(\zeta_i^{(k-1)} +$$
$$+ D)\Delta\zeta_i^{(k-1)}\zeta_i^{(k-1)};$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$S_n^{(k)} = (\zeta_i^{(k-1)} + D)^{n-2}\Delta\zeta_i^{(k-1)}\zeta_i^{(k-1)};$$

$R^{(k)}$ is the sum of terms constituting product of coefficients depending on magnitude $\eta_{i,k+1}$ and $H_{i,k+1}$ and whole powers (no lower than second) of magnitude $\Delta r_i^{(k-1)}$.

We will present sum $S_1^{(k)} + S_2^{(k)} + \ldots + S_n^{(k)}$ in the form

$$S_1^{(k)} + S_2^{(k)} + \ldots + S_n^{(k)} = S_*^{(k)}\Delta\zeta_i^{(k-1)}.$$

where $S_\Delta^{(k)}$ is a certain function of magnitudes $\eta_{i,k+1}$ and t.

Asymptotic form of functions $S_0^{(k)}$, $S_\Delta^{(k)}$ and $R^{(k)}$ during $t \to \infty$ we will determine by equalities

$$\left.\begin{aligned} S_0^{(k)} &= M_0^{k}\left[t^{\mu_k-1} + o(t^{\mu_k-1})\right], \\ S_\Delta^{(k)} &= M_\Delta^{(k)}\left[t^{\mu_\Delta^{(k)}} + o\left(t^{\mu_\Delta^{(k)}}\right)\right], \\ R^{(k)} &= M_R^{(k)}\left[t^{\mu_R^{(k)}} + o\left(t^{\mu_R^{(k)}}\right)\right]. \end{aligned}\right\} \tag{8.97}$$

We will designate

$$\left.\begin{aligned} m(\mu_\Delta^{(k)}) &= \max_{H_{i,k+1},\, \eta_{i,k+1}} \mu_\Delta^{(k)}, \\ m(\mu_R^{(k)}) &= \max_{\eta_{i,k+1},\, H_{i,k+1}} \mu_R^{(k)}. \end{aligned}\right\} \tag{8.98}$$

Assuming that during $t \to \infty$

$$R^{(k)} = o(S_0^{(k)}). \tag{8.99}$$

we will determine magnitude $\eta_{i,k+1}$ from condition

$$S_0^{(k)} + S_\Delta^{(k)} \Delta \zeta_i^{(k-1)} = o(S_0^{(k)}). \tag{8.100}$$

This condition leads to a formula for determining coefficient $\eta_{i,k+1}$, expressed by first equality of system (8.93).

If coefficient $M_\Delta^{(k)}$ with this value of coefficient $\eta_{i,k+1}$ turns into zero or if $m(\mu_\Delta^{(k)}) \neq \mu_\Delta^{(k)}$, then condition (8.100) is not executed. Such a case we will consider special, which does not allow application of considered method for determining unknown coefficients.

The case when

$$M_\Delta^{(k)} \neq 0 \tag{8.101}$$

during value of coefficient $\eta_{i,k+1}$, determined by above-indicated method, we will call <u>model</u>. In this case, condition (8.100) is executed if coefficient $H_{i,k+1}$ is determined by second equality of system (8.93).

We will compare recurrence dependences (8.93) with formulas (8.86) and (8.87) for determination of coefficients η_{i2} and H_{i2}. It is easy to note that dependence, expressed by formulas (8.86) and (8.87), it is possible to consider as a particular case of dependences (8.93), if one extends region of application of the latter to case k = 1. Thus, <u>dependences (8.93) determine a single method of finding coefficients η_{ij} and H_{ij}</u> for all $j > 1$.

The above-mentioned logical foundation of the considered method of finding coefficients η_{ij} and H_{ij} during $j > 2$ rests on two assumptions:

$$a)\ \eta_{l,\,k+1} < \eta_{lk}. \qquad (8.96)$$

$$b)\ R^{(k)} = o(S_0^{(k)}). \qquad (8.99)$$

Inasmuch as these assumptions were used by us only for establishment of the mentioned method and this problem is already solved, the necessity for proof of their validity from this side does not appear. However, the correctness of the found method is in straight dependence on fulfillment of condition (8.96), since in accordance with § 2, we are interested only in such approximate solutions of an equation of oscillations, which satisfy this condition. Therefore, it is necessary to be convinced of the fact that condition (8.96) is executed and that with this $\mu_{k-1} < \mu_{k-2}$.

Since during proof of inequality (8.96) there plays an essential role the first of equalities (8.93), in which appears magnitude $m(\mu_\Delta^{(k)})$, then we will make certain preliminarily searches into parts of the character and numerical determination of this magnitude. Results, which will be obtained below, may also be used as definitizing indications when determining, by the given method, coefficients η_{ij} and H_{ij}.

We will start from magnitude $m(\mu_\Delta^{(1)})$.

Considering determination (8.84) and after considering formula (8.81), we will find

$$\begin{aligned} m\left(\mu_\Delta^{(1)}\right) = \max\,[&(n-1)\eta_{l1},\ \max(\beta_1,\ -1) + \\ +\ &(n-2)\eta_{l1},\ \max(\beta_2,\ \beta_1-1,\ -2) + (n-3)\eta_{l1}, \cdots \\ \cdots,\ &\max(\beta_{n-2},\ \beta_{n-1}-1, \cdots, \beta_1-n+3,\ 2-n) + \\ +\ &\eta_{l1},\ \max(\beta_{n-1},\ \beta_{n-2}-1, \cdots, \beta_1-n+2,\ 1-n)] \end{aligned} \qquad (8.102)$$

This formula needs the following explanation: if $b_j = 0$, then symbol β_j one should delete (this is valid for any j).

Now we will establish formula for magnitude $m(\mu_\Delta^{(k)})$ during arbitrary $k > 1$.

We will compare magnitudes $S_j^{(k)}$ and $S_j^{(1)}$ ($j = 1, \ldots, n$). After presenting magnitude $\zeta_i^{(k)}$ in the form

$$\zeta_i^{(k)} = \zeta_i^{(0)} + \sum_{l=0}^{k-1} \Delta\zeta_i^{(l)}$$

and converting magnitude $S_j^{(k)}$ ($j = 1, \ldots, n$) analogously to how magnitudes $S_j^{(1)}$ ($j = 1, \ldots, n$), were converted, we will find that magnitude $S_\Delta^{(k)}$ may be expressed in the form

$$S_\Delta^{(k)} = S_{\Delta 0}^{(k)} + \Delta S_\Delta^{(k)}.$$

where $S_{\Delta 0}^{(k)}$ is determined by the formula which is obtained from the formula for determination of magnitude $S_{\Delta}^{(1)}$ after replacement of magnitude η_{12} by magnitude $\eta_{i,k+1}$ and $\Delta S_{\Delta}^{(k)}$ is a certain additional exponential or compound-exponential function of t.

Because of the fact that exponent t for any magnitude $\Delta\zeta_i^{(j)}$ $(j = 0, \ldots, k - 1)$ is less than exponent t for magnitude $\zeta_i^{(0)}$, in a model case during $t \to \infty$

$$\Delta S_{\Delta}^{(k)} = o(S_{\Delta 0}^{(k)}).$$

Consequently, in a model case magnitude $m(\mu_{\Delta}^{(k)})$ is exponent t for exponential function, asymptotically equivalent during $t \to \infty$ to magnitude $S_{\Delta 0}^{(k)}$. But since formulas for determining magnitudes $S_{\Delta}^{(1)}$ and $S_{\Delta 0}^{(k)}$ there exists the above-indicated connection, and magnitude $m(\mu_{\Delta}^{(1)})$ does not depend on magnitude η_{12}, then magnitude $m(\mu_{\Delta}^{(k)})$ does not depend on magnitude $\eta_{i,k+1}$ and, consequently, is equal to first magnitude, i.e.,

$$m(\mu_{\Delta}^{(k)}) = m(\mu_{\Delta}^{(1)}). \tag{8.103}$$

Passing to proof of inequalities (8.96) and $\mu_{k-1} < \mu_{k-2}$, let us note that because of the principle of full mathematical induction they can be considered proven if the two following affirmations are proven:

a) if there is executed inequality

$$\eta_{ik} < \eta_{i,k-1},$$

then there is executed inequality

$$\mu_{k-1} < \mu_{k-2}; \tag{8.104}$$

b) if there is executed inequality (8.104), then there is executed inequality (8.96).

For proof of the first affirmation we will compare magnitudes

$$(\zeta_i^{(k-1)} + D)^{n-1}\zeta_i^{(k-1)} + b_1(\zeta_i^{(k-1)} + D)^{n-2}\zeta_i^{(k-1)} + \ldots + b_n$$

and

$$(\zeta_i^{(k-2)} + D)^{n-1}\zeta_i^{(k-2)} + b_1(\zeta_i^{(k-2)} + D)^{n-2}\zeta_i^{(k-2)} + \ldots + b_n,$$

which we will present, correspondingly, in the form of sums

$$(\zeta_i^{(k-1)})^n + \binom{n}{1}(\zeta_i^{(k-1)})^{n-1}D\zeta_i^{(k-1)} + \ldots + D^{n-1}\zeta_i^{(k-1)} + \ldots + b_n$$

and

$$(\zeta_i^{(k-2)})^n + \binom{n}{1}(\zeta_i^{(k-2)})^{n-1}D\zeta_i^{(k-2)} + \ldots + D^{n-1}\zeta_i^{(k-2)} + \ldots + b_n.$$

The difference of each pair of corresponding components of these sums (with the exception of the difference of the last pair, which is equal to zero), in accordance with formulas

$$\left.\begin{array}{l}
\zeta_i^{(k-1)}=\zeta_i^{(k-2)}+\Delta\zeta_i^{(k-2)}, \\
\dfrac{d\zeta_i^{(k-1)}}{dt}=\dfrac{d\zeta_i^{(k-2)}}{dt}+\eta_{ik}\Delta\zeta_i^{(k-2)}t^{-1}, \\
\dots\dots\dots\dots\dots\dots \\
\dfrac{d^{n-1}\zeta_i^{(k-1)}}{dt^{n-1}}=\dfrac{d^{n-1}\zeta_i^{(k-2)}}{dt^{n-1}}+ \\
+(\eta_{ik}-1)\dots(\eta_{ik}-n+2)\Delta\zeta_i^{(k-2)}t^{1-n},
\end{array}\right\} \tag{8.105}$$

may be located in a finite series in whole, increasing (from unity) powers of magnitude $\Delta\zeta_i^{k-2}$. Let us assume that inequality

$$\eta_{ik}<\eta_{i,k-1}$$

is executed. Then exponents of exponential functions, asymptotically equivalent during $t\rightarrow\infty$ to the second and following components of each such sum, will be less than exponents of exponential functions, asymptotically equivalent to their first component, and, consequently, less than magnitude

$$m(\mu_k^{(k-1)})+\eta_{ik}.$$

Therefore during $t\rightarrow\infty$

$$R^{(k-1)}=o(S_k^{(k-1)}\Delta\zeta_i^{(k-2)})=o(S_0^{(k-1)}).$$

Hence because of asymptotic equivalence of magnitudes $S_0^{(k-1)}$ and $-S_k^{(k-1)}\Delta\zeta_i^{(k-2)}$, established by the taken method of determining coefficients $\eta_{i,j}$ and $H_{i,j}$, applied to coefficients η_{ik} and H_{ik}, there follows

$$S_0^{(k-1)}=o(S_0^{(k)}),$$

which leads to inequality (8.104).

Proof of the second affirmation is significantly simpler.

Since in the taken method of determining coefficients

$$\mu_{k-2}=m(\mu_k^{(k-1)})+\eta_{ik}$$

and

$$\mu_{k-1}=m(\mu_k^{(k)})+\eta_{i,k+1},$$

then because of equality (8.103)

$$\eta_{i,k+1}-\eta_{i,k}=\mu_{k-1}-\mu_{k-2}.$$

Consequently, if $\mu_{k-1}<\mu_{k-2}$, then also $\eta_{i,k+1}<\eta_{i,k}$, as it was required to prove.

Example: We will determine coefficients η_{i1} and η_{i2} ($i = 1, 2, 3$) for an equation of free oscillations

$$\dddot{x} + ct^{\sigma}x = 0, \quad c \neq 0. \tag{8.106}$$

As was shown in § 3, ways of determining coefficients are different depending upon asymptotic properties of coefficients $b_j^{(1)}$ which are connected with coefficients b_j by formulas (8.19). In this case we have

$$b_1^{(1)} = b_2^{(1)} = 0, \quad b_3^{(1)} = ct^{\sigma+3}.$$

Let us consider at first the case $\sigma < -3$, i.e., a case obeying equalities (8.27).

As was shown in § 3, here for all i

$$\eta_{i1} \leq -1,$$

where value $\eta_{i1} = -1$ corresponds to two values of coefficient H_{i1}:

$$H_{11} = 1 \text{ and } H_{21} = 2.$$

The third pair of values of coefficients η_{i1} and H_{i1} we will find by formulas (8.32). We have

$$\eta_{31} = \sigma + 2, \quad H_{31} = \frac{-c}{(\sigma + 2)(\sigma + 1)}.$$

Now let us turn to case $\sigma = -3$.

In this case, as was shown in § 3, also for all i there is executed inequality

$$\eta_{i1} \leq -1.$$

Since

$$\lim_{t \to \infty} b_3^{(1)} = c,$$

there is executed inequality (8.35). Consequently, in this case for all i

$$\eta_{i1} = -1,$$

and coefficients H_{11}, H_{21}, H_{31} are roots of equation

$$x_i^3 - 3x_i^2 + 2x_i + c = 0. \tag{8.107}$$

If

$$c^2 < \frac{4}{27},$$

then all roots of this equation are real. Applying formula for determining roots of cubic equation [27], we will find

$$H_{11} = \frac{2}{\sqrt{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{3c\sqrt{3}}{2}\right)\right] + 1, \quad H_{21} = \frac{2}{\sqrt{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{3c\sqrt{3}}{2}\right) + \frac{2\pi}{3}\right] + 1,$$

$$H_{31} = \frac{2}{\sqrt{3}} \cos\left[\frac{1}{3} \arccos\left(-\frac{3c\sqrt{3}}{2}\right) + \frac{4\pi}{3}\right] + 1.$$

If

$$c^2 = \frac{4}{27},$$

then all roots also are real, but among them is two multiples. During

$$c = \frac{2}{3\sqrt{3}}$$

these multiple roots are equal to magnitude

$$1 + \frac{1}{\sqrt{3}};$$

during

$$c = -\frac{2}{3\sqrt{3}}$$

multiple roots are equal to magnitude

$$1 - \frac{1}{\sqrt{3}}.$$

The third root in the first case is equal to magnitude

$$1 - \frac{2}{\sqrt{3}},$$

into second case, to magnitude

$$1 + \frac{2}{\sqrt{3}}.$$

Thus, during

$$c = \pm \frac{2}{3\sqrt{3}}$$

we have

$$H_{11} = H_{21} = 1 \pm \frac{1}{\sqrt{3}},$$

$$H_{31} = 1 \mp \frac{2}{\sqrt{3}}.$$

If

$$c^2 > \frac{4}{27},$$

then equation (8.106) has one real and two complex roots. Determining them, we will find

$$H_{11} = \sqrt[3]{A+B} + \sqrt[3]{A-B} + 1,$$

$$H_{21} = -\frac{\sqrt[3]{A+B} + \sqrt[3]{A-B}}{2} + i\frac{\sqrt{3}\left(\sqrt[3]{A+B} - \sqrt[3]{A-B}\right)}{2} + 1,$$

$$H_{31} = -\frac{\sqrt[3]{A+B} + \sqrt[3]{A-B}}{2} - i\frac{\sqrt{3}\left(\sqrt[3]{A+B} - \sqrt[3]{A-B}\right)}{2},$$

where

$$A = -\frac{c}{2},$$

$$B = \sqrt{\frac{c^2}{4} - \frac{1}{27}},$$

$$i = \sqrt{-1}.$$

Let us consider now the last possible case, $\sigma > -3$.

This case is a particular realization of a case in which some of the coefficients of equation (0.1) satisfy condition (8.41).

We will apply substitution (8.42), where in accordance with equality (8.46)

$$\xi = 1 \cdot \frac{\sigma}{3},$$

and will determine coefficients η'_{i1} and H'_{i1} ($i = 1, 2, 3$).

Since in this case $k = 0$ (see page 239), then for all i we will obtain

$$\eta'_{i1} = -1.$$

Coefficients H'_{i1} ($i = 1, 2, 3$) are roots of equation (8.54). In the considered case

$$\lim_{t\to\infty} c_1^{(i)} = \lim_{t\to\infty} c_2^{(i)} = 0,$$
$$\lim_{t\to\infty} c_3^{(i)} = c.$$

Therefore,

$$\left.\begin{aligned} H'_{11} &= \sqrt[3]{c}, \\ H'_{21} &= \sqrt[3]{c}\left(\frac{-1+i\sqrt{3}}{2}\right), \\ H'_{31} &= \sqrt[3]{c}\left(\frac{-1-i\sqrt{3}}{3}\right). \end{aligned}\right\} \qquad (8.108)$$

Because of equality (8.49), values of these coefficients coincide with values of corresponding coefficients H_{i1}. Considering equality (8.50), we will obtain for all coefficients η_{i1} value

$$\eta_{i1} = \frac{\sigma}{3}.$$

Found values of coefficients η_{i1} and H_{i1} ($i = 1, 2, 3$) in all considered cases correspond to approximate solutions

$$\exp \int H_{i1} t^{\eta_{i1}}\, dt.$$

In cases $\sigma = -3$ and $\sigma = 0$ they coincide with exact solutions.

We will determine coefficients η_{i1} and H_{i2} for case $\sigma > -3$.

In accordance with formulas given in this paragraph, for magnitudes $S_0^{(1)}$ and $S_\Delta^{(1)}$ we will obtain

$$\left.\begin{aligned} S_0^{(1)} &= (\zeta_i^{(0)})^3 + 3\dot{\zeta}_i^{(0)}\zeta_i^{(0)} + \ddot{\zeta}_i^{(0)} + c = H_{i1}^3 t^\sigma + \sigma H_{i1}^2 t^{\frac{2\sigma}{3}-1} + H_{i1}\frac{\sigma(\sigma-3)}{9} t^{\frac{\sigma}{3}-2} + ct^\sigma = \\ &= \sigma H_{i1} t^{\frac{\sigma}{3}-1}\left(H_{i1} t^{\frac{\sigma}{3}} + \frac{\sigma-3}{9} t^{-1}\right). \\ S_\Delta^{(1)} &= 3(\zeta_i^{(0)})^2 + 3(\eta_{i1} + \eta_{i2}) t^{-1}\zeta_i^{(0)} + \eta_{i2}(\eta_{i2}-1) t^{-2} = \\ &= 3H_{i1}^2 t^{\frac{2\sigma}{3}} + \left(\frac{2\sigma}{3} + 2\eta_{i2}\right) H_{i1} t^{\frac{\sigma}{3}-1} + \eta_{i2}(\eta_{i2}-1) t^{-2} = \\ &= H_{i1} t^{\frac{\sigma}{3}}\left(3H_{i1} t^{\frac{\sigma}{3}} + \frac{2\sigma}{3} + 2\eta_{i2} t^{-1}\right) + \eta_{i2}(\eta_{i2}-1) t^{-2} \quad (i = 1, 2, 3). \end{aligned}\right\} \qquad (8.109)$$

Since

$$\frac{s}{3} > -1,$$

then from the first equalities we will obtain

$$\gamma_{i1} = \frac{2s}{3} - 1.$$

and from the second

$$m(\mu_2^{(1)}) = \frac{2s}{3}.$$

By formula (8.86) we will find

$$\gamma_{i2} = \frac{2s}{3} - 1 - \frac{2s}{3} = -1 \qquad (i = 1, 2, 3).$$

By formula (8.87) we will find

$$-H_{i2} = \frac{sH_{i1}^2}{3H_{i1}^2} = \frac{s}{3} \qquad (i = 1, 2, 3).$$

In accordance with the obtained results we will write the formula for three approximate solutions of equation (8.106):

$$\tilde{x}_i(t) = C_i \exp \int_T^t (H_{i1} t^{\gamma_{i1}} + H_{i2} t^{\gamma_{i2}})\, dt =$$

$$= C_i' \left(\frac{T}{t}\right)^{\frac{s}{3}} \exp \frac{3H_{i1}\left(t^{\frac{s}{3}+1} - T^{\frac{s}{3}+1}\right)}{s+3} \qquad (i = 1, 2, 3). \tag{8.110}$$

where C_i and C_i' ($i = 1, 2, 3$) are arbitrary constants, and coefficients H_{i1} ($i = 1, 2, 3$) are determined by formulas (8.108).

§ 5. Asymptotic Properties of Free Oscillations

Asymptotic properties of free oscillations, presented by equations of the class considered in this chapter, can be investigated by general methods given in Chapters V and VI. With this goal, there can be used both canonical expansions of the solution of an equation of free oscillations, considered in Chapter II and canonical expansions in which are used functions

$$\zeta_1^{(k)}(t), \ldots, \zeta_n^{(k)}(t) \text{ вида } (8.94)$$

$$\zeta_i^{(k)}(t) = \sum_{j=1}^{k+1} H_{ij} t^{\gamma_{ij}}.$$

All the theory of canonical expansions connected with functions $\zeta_1^{(k)}(t), \ldots, \zeta_n^{(k)}(t)$, with the exception of the conditions of their applicability, and, founded on this theory, the methods of research of asymptotic properties of free oscillations during the new method, considered here, of determining functions $\zeta_i^{(k)}(t)$, remains in force.

Sufficient condition of applicability of k-th canonical expansion of unmodulated structure, in this case, will lead to the requirement that determinant

$$\begin{vmatrix} 1 & \dots & 1 \\ \zeta_1^{(k-1)} & \dots & \zeta_n^{(k-1)} \\ \dots & \dots & \dots \\ (\zeta_1^{(k-1)} + D)^{n-2}\zeta_1^{(k-1)} & \dots & (\zeta_n^{(k-1)} + D)^{n-2}\zeta_n^{(k-1)} \end{vmatrix} \tag{8.111}$$

be different than zero during sufficiently large values of t.

If this condition is not executed, then in a number of cases there can be built canonical expansions of modified structure according to methods shown in § 5, Chapter II, with that only distinction being that functions $\zeta_1^{(k-1)}(t)$, ..., $\zeta_n^{(k-1)}(t)$ in this case correspond to determination (8.94).

Determining functions $\zeta_i^{(k-1)}(t)$ in the form of (8.94) allows us, in many cases, very effectively to use certain general results obtained in Chapters V and VI. Such a possibility is connected with the tendency toward lowering during growth of k exponents of functions asymptotically equivalent during $t \to \infty$ to functions $h_{i,j}^{(k)}(t)$.

For confirmation of the fact that the mentioned exponents in very general cases indeed have a tendency toward lowering, we will consider sequence of sets of functions

$$\zeta_1^{(k-1)}(t), \dots, \zeta_n^{(k-1)}(t) \qquad (k = 1, 2, \dots), \tag{8.112}$$

the first element of which satisfies condition: coefficient during highest power t of polynomial

$$W_1(t) = \begin{vmatrix} 1 & \dots & 1 \\ \zeta_1^{(0)} & \dots & \zeta_n^{(0)} \\ (\zeta_1^{(0)} + D)\zeta_1^{(0)} & \dots & (\zeta_n^{(0)} + D)\zeta_n^{(0)} \\ \dots & \dots & \dots \\ (\zeta_1^{(0)} + D)^{n-2}\zeta_1^{(0)} & \dots & (\zeta_n^{(0)} + D)^{n-2}\zeta_n^{(0)} \end{vmatrix}$$

is different than zero.

Obviously, if there is executed shown condition, then the addition to functions $\zeta_i^{(0)}(t)$ (i = 1, ..., n) of exponential functions with smaller exponents as components will not change the asymptotic form of polynomial $W_1(t)$. We will designate exponent of function, asymptotically equivalent to this polynomial, by symbol η_W. Because of that said, all polynomials $W_k(t)$ also are equivalent to exponential functions with this exponent.

We will designate by symbols μ_{wi} the highest exponents of polynomials from t

$$(\zeta_i^{(0)} + D)^{n-2}\zeta_i^{(0)} \qquad (i = 1, \dots, n).$$

independently of the fact that coefficients with these powers are different or equal to zero. Then from assumed property of polynomial $W_1(t)$, it follows that adjoints $w_{ni}^{(1)}(t)$ during $t \to \infty$ are asymptotically equivalent to exponential functions whose exponents do not exceed magnitudes

$$\eta_0 - \mu_{1i} \ (i = 1, \ldots, n).$$

During transition to second and following elements of sequence of sets of functions (8.112), this property for adjoint $w_{ni}^{(k)}(t)$ is kept.

Exponent of exponential function, asymptotically equivalent to magnitude

$$(\zeta_i^{(k-1)} + D)^{n-1}\zeta_i^{(k-1)} + b_1(\zeta_i^{(k-1)} + D)^{n-2}\zeta_i^{(k-1)} + \ldots + b_n,$$

in the preceding paragraph was designated by symbol μ_{k-1}. Since this index depends on selected function $\zeta_i^{(k-1)}(t)$, i.e., on index i, we will definitize this designation after replacing symbol μ_{k-1} by symbol $\mu_{k-1}(i)$.

We will turn to formula for coefficients $h_{ij}^{(k)}$ [see § 4 Chapter II, explanation to equations (2.49)]. After comparing this formula with results obtained here, we will find that coefficient $h_{ij}^{(k)}$ during $t \to \infty$ is asymptotically equivalent to exponential function whose exponent does not exceed magnitude

$$\mu_{k-1}(j) - \mu_{ki}.$$

But according to principle of construction of functions $\zeta_i^{k-1}(t)$ $(k = 1, 2, \ldots)$, coefficient $\mu_{k-1}(j)$ decreases with growth of k. Therefore, with growth of k, the mentioned exponent is limited from above by a smaller and smaller magnitude, having, thus, a tendency toward lowering.

In connection with the considered tendency for change in exponents of functions asymptotically equivalent to functions $h_{ij}^{(k)}(t)$, in many cases there appears the possibility during a certain sufficiently large number k, to obtain as rapidly diminishing or as slowly growing functions $h_{ij}^{(k)}(t)$ as is required by the conditions of application for one or another theorem or appraisal.

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